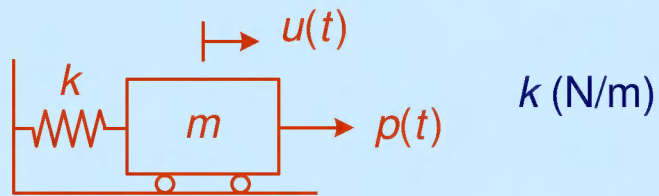


SDF - INTRODUCTION

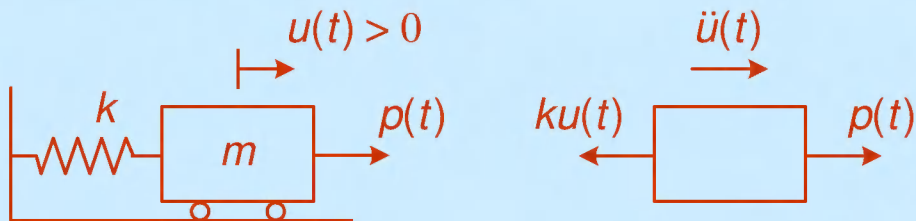
A single degree of freedom system consisting of a mass m and a spring with stiffness k is considered.



The spring is undeformed for $u = 0$

The equation of motion can be derived in two ways.

Newton's second law

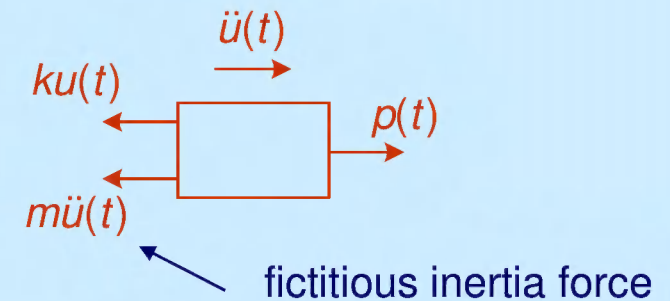


$$m \ddot{u}(t) = p(t) - k u(t) \quad \rightarrow \quad \boxed{m \ddot{u} + k u = p(t)}$$

D'Alembert's principle

The system is supposed in dynamic equilibrium. The principles of statics are applied by introducing a fictitious inertia force, a force equal to the product of mass time its acceleration and acting in a direction opposite to the acceleration.

Free-body diagram



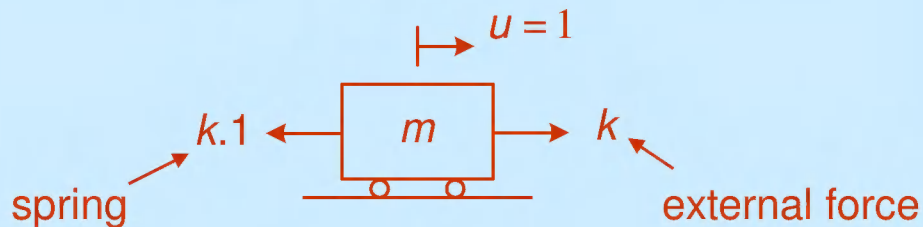
Dynamic equilibrium

$$p(t) - k u(t) - m \ddot{u}(t) = 0$$

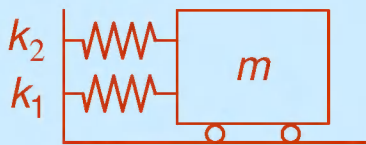
Stiffness k

The stiffness k has the same definition as in the displacement method:

The stiffness k is the external force that is needed to keep the system in equilibrium when a unit displacement $u = 1$ is applied.



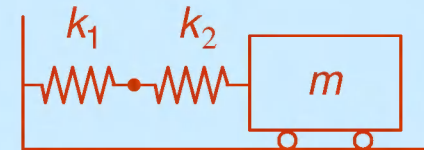
Combination of two springs – case 1



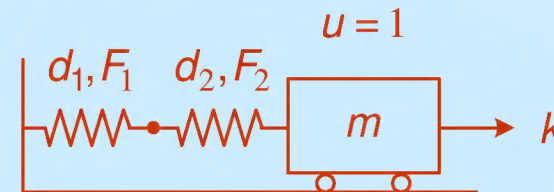
The stiffness for this system is trivial:

$$k = k_1 + k_2$$

Combination of two springs – case 2



This case is more complicated.



$$F_1 = k_1 d_1$$

$$F_2 = k_2 d_2$$

$$\text{statics: } F_1 = F_2 = k$$

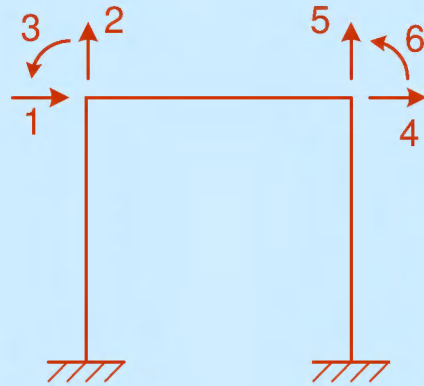
$$d_1 + d_2 = 1 \rightarrow \frac{k}{k_1} + \frac{k}{k_2} = 1$$

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

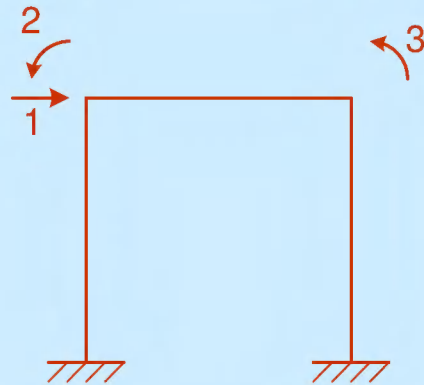
Some structures can be idealised as SDF

Example

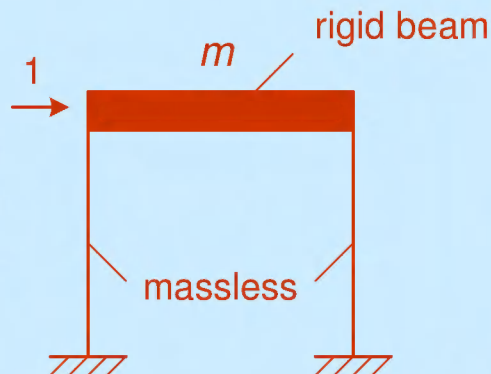
In statics, this frame has 6 active degrees of freedom.



By neglecting the axial deformations, 3 d.o.f. disappear.

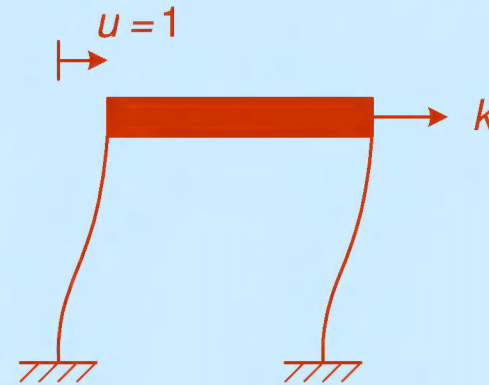


Only one d.o.f. is left if the frame is consisting of an heavy roof supported by light columns.

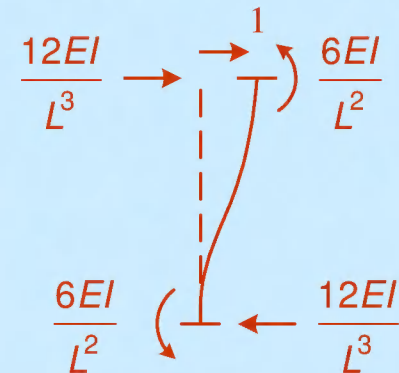


The mass of this SDF system is m , the mass of the roof.

The stiffness is determined in the classical way:

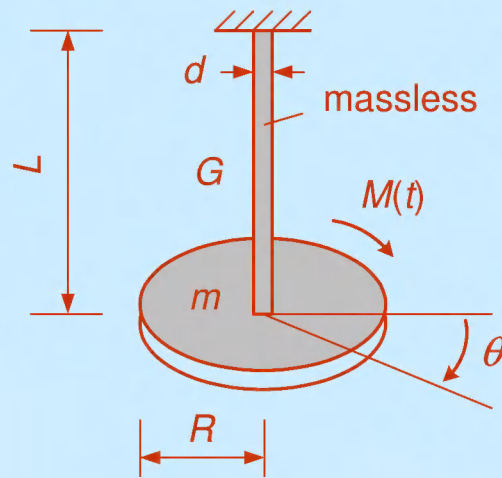


$$k = \frac{24EI}{L^3}$$



Rotations

A SDF system can also have a rotational movement.



in statics: $M = C \theta$ $C = \frac{\pi d^4 G}{32 L}$

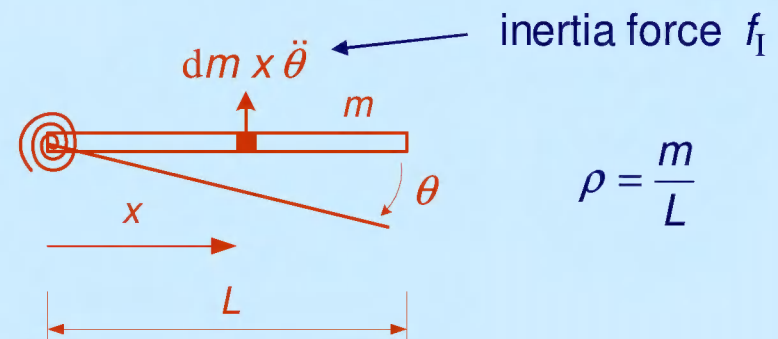
in dynamics: $J \ddot{\theta} + C \theta = M(t)$

J is the moment of inertia.

for a circular disk: $J = \frac{mR^2}{2}$

$J \ddot{\theta}$ is the moment created by the fictitious inertia forces.

Example: Calculation of J for a bar



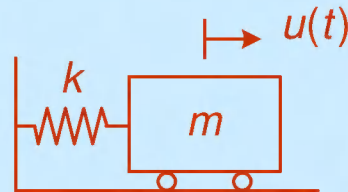
$$J \ddot{\theta} = \sum f_I \cdot x = \int_0^L x^2 dm \cdot \ddot{\theta}$$

$$J = \int_0^L x^2 dm = \rho \int_0^L x^2 dx = \rho \frac{L^3}{3} = \frac{mL^2}{3}$$

SDF - FREE VIBRATION

UNDAMPED FREE VIBRATION

The structure is disturbed from its static equilibrium and then vibrates without any applied forces.



The equation of motion is:

$$m \ddot{u} + k u = 0$$

The solution is: $u(t) = A \cos(\omega_n t) + B \sin(\omega_n t)$

$$\omega_n = \sqrt{k/m} \quad (\text{rad/s}) \quad \text{natural circular frequency}$$

A and B are determined by the *initial conditions*

$$u_{t=0} = u_0 \rightarrow u_0 = A$$

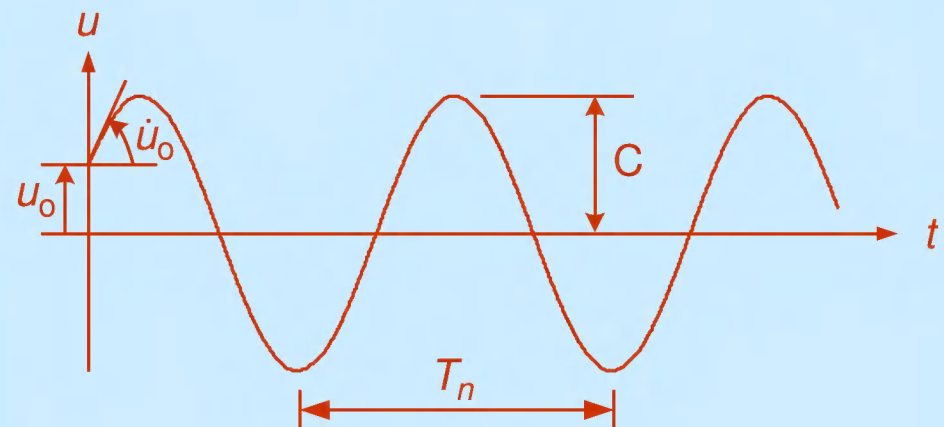
$$\dot{u}_{t=0} = \dot{u}_0 \rightarrow \dot{u}_0 = B \omega_n$$

$$u(t) = u_0 \cos(\omega_n t) + \frac{\dot{u}_0}{\omega_n} \sin(\omega_n t)$$

which can be written as

$$u(t) = C \sin(\omega_n t + \theta)$$

$$C = \sqrt{u_0^2 + (\dot{u}_0/\omega_n)^2} \quad \cos \theta = \frac{\dot{u}_0/\omega_n}{C} \quad \sin \theta = \frac{u_0}{C}$$



natural period

$$T_n = \frac{2\pi}{\omega_n} \quad (\text{s})$$

natural frequency

$$f_n = \frac{1}{T_n} = \frac{\omega_n}{2\pi} \quad (\text{Hz})$$

Energy in undamped free vibration

At each instant of time, the total energy E is made of two parts, the kinetic energy E_k and the strain energy E_s .

$$E_s(t) = \frac{1}{2} k u(t)^2$$

$$E_k(t) = \frac{1}{2} m \dot{u}(t)^2$$

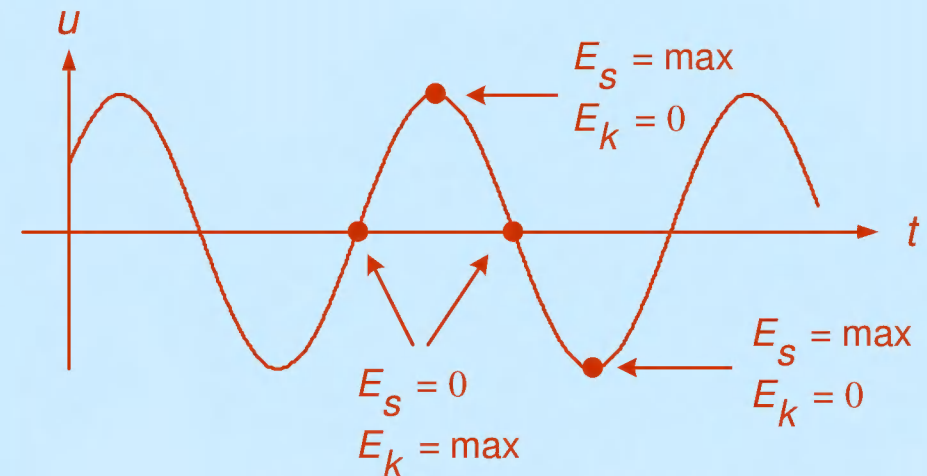
$$E(t) = E_k(t) + E_s(t)$$

$$= \frac{1}{2} m [C \omega_n \cos(\omega_n t + \theta)]^2 + \frac{1}{2} k [C \sin(\omega_n t + \theta)]^2$$

$$= \frac{1}{2} m C^2 \omega_n^2 \cos^2(\omega_n t + \theta) + \frac{1}{2} k C^2 \sin^2(\omega_n t + \theta)$$

$$= \frac{1}{2} k C^2 \quad (k = m \omega_n^2)$$

$E(t)$ is constant, which implies conservation of energy.



Remark : the conservation of energy can be used to derive the differential equation.

$$E(t) = \frac{1}{2} m \dot{u}^2 + \frac{1}{2} k u^2$$

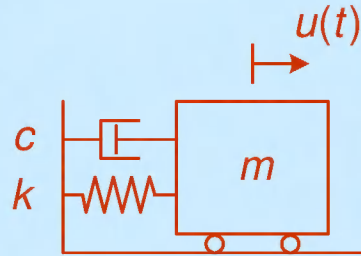
conservation of energy $\rightarrow \frac{dE}{dt} = 0$

$$\frac{dE}{dt} = m \dot{u} \ddot{u} + k u \dot{u} = 0 \rightarrow m \ddot{u} + k u = 0$$

VISCOUSLY DAMPED FREE VIBRATION

Friction in the structure is idealized by a linear viscous damper which develops a force proportional to the velocity

$$f_D = -c \dot{u}(t)$$



The equation of motion is:

$$m \ddot{u} + c \dot{u} + k u = 0$$

if $c < c_r = 2\sqrt{km}$ (critical damping) the solution is

$$u(t) = e^{-\xi \omega_n t} [A \cos(\omega_D t) + B \sin(\omega_D t)]$$

damping ratio

$$\xi = \frac{c}{c_r} = \frac{c}{2\sqrt{km}}$$

damped pulsation

$$\omega_D = \omega_n \sqrt{1 - \xi^2}$$

A and B are determined by the *initial conditions*.

$$\begin{aligned} \dot{u}(t) = & -\xi \omega_n e^{-\xi \omega_n t} [A \cos(\omega_D t) + B \sin(\omega_D t)] \\ & + e^{-\xi \omega_n t} [-A \omega_D \sin(\omega_D t) + B \omega_D \cos(\omega_D t)] \end{aligned}$$

$$u_{t=0} = u_0 \rightarrow u_0 = A$$

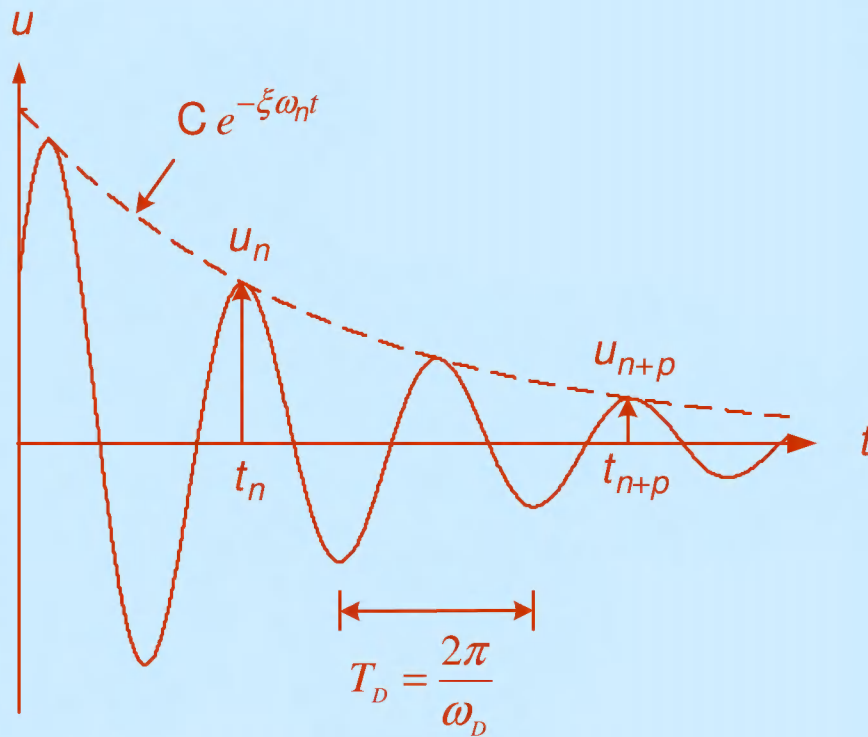
$$\dot{u}_{t=0} = \dot{u}_0 \rightarrow \dot{u}_0 = -\xi \omega_n A + B \omega_D$$

$$u(t) = e^{-\xi \omega_n t} \left[u_0 \cos(\omega_D t) + \frac{\dot{u}_0 + \xi \omega_n u_0}{\omega_D} \sin(\omega_D t) \right]$$

The solution can also be written as

$$u(t) = C e^{-\xi \omega_n t} \sin(\omega_D t + \theta) \quad \sin \theta = \frac{u_0}{C}$$

$$C = \sqrt{u_0^2 + \left(\frac{\dot{u}_0 + \xi \omega_n u_0}{\omega_D} \right)^2} \quad \cos \theta = \frac{\dot{u}_0 + \xi \omega_n u_0}{C \omega_D}$$



Decay of motion

A free vibration test can be used to determine experimentally the natural frequency and the damping of a structure.

p periods between two maximal points u_n and u_{n+p}

$$t_{n+p} = t_n + pT_D$$

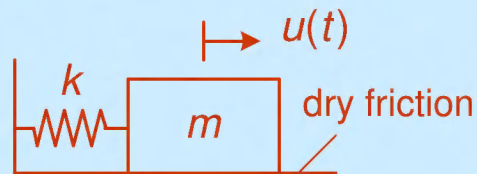
$$\begin{aligned} \frac{u_n}{u_{n+p}} &= \frac{C e^{-\xi \omega_n t_n} \sin(\omega_D t_n + \theta)}{C e^{-\xi \omega_n (t_n + pT_D)} \sin(\omega_D (t_n + pT_D) + \theta)} \\ &= e^{\xi \omega_n p T_D} \end{aligned}$$

$$\ln \frac{u_n}{u_{n+p}} = \xi \omega_n p T_D = \xi \omega_n p \frac{2\pi}{\omega_n \sqrt{1-\xi^2}}$$

$$\xi < 0.1 \rightarrow \sqrt{1-\xi^2} \approx 1 \rightarrow \boxed{\xi = \frac{1}{2\pi p} \ln \frac{u_n}{u_{n+p}}}$$

COULOMB-DAMPED FREE VIBRATION

Coulomb damping results from friction against sliding of two dry surfaces.



The friction force is $F = \mu N$ where μ denotes the coefficients of static and kinetic friction, taken to be equal, and N the normal force between the sliding surfaces.

F is assumed to be independent of the velocity of the motion and its direction opposes motion.

The equations of motion from left to right are

$$m \ddot{u} + k u = -F$$

$$u(t) = A_1 \cos(\omega_n t) + B_1 \sin(\omega_n t) - F/k$$

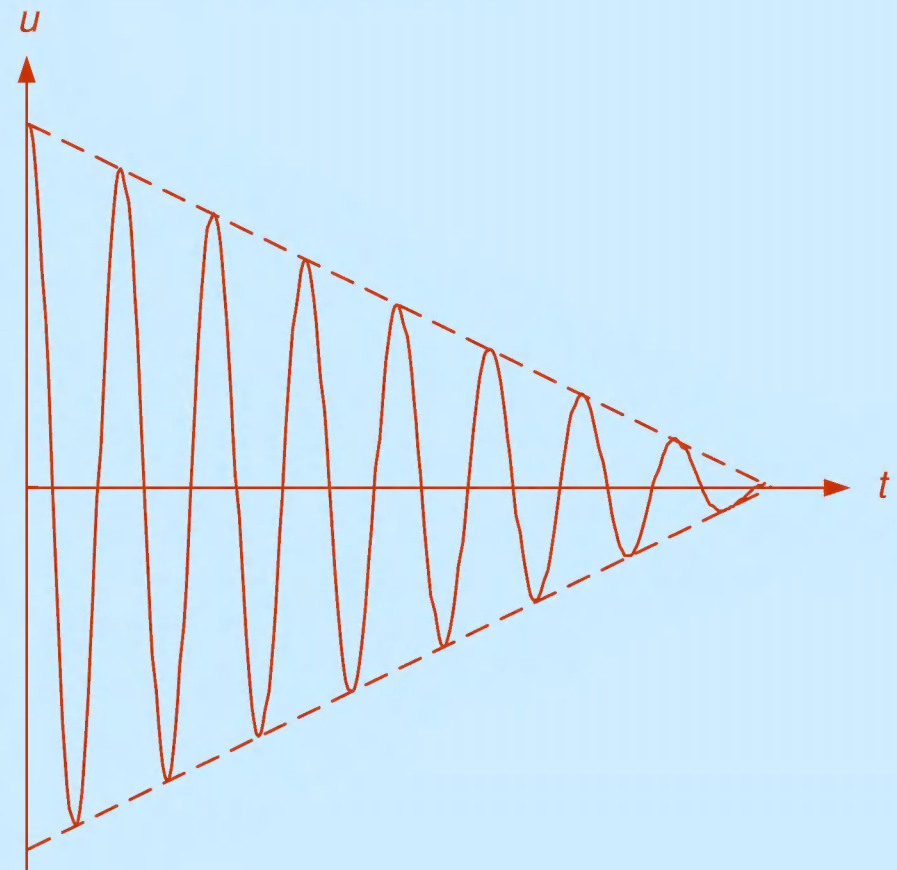
The equations of motion from right to left are

$$m \ddot{u} + k u = F$$

$$u(t) = A_2 \cos(\omega_n t) + B_2 \sin(\omega_n t) + F/k$$

The constants A_1, B_1, A_2, B_2 depend on the initial conditions of each successive half-cycle motion.

The plot of the solution is



DAMPING

Different damping models can be used, a viscous proportional damping is the most used approach.

There are two reasons for that:

- The mathematical equation which describes the motion is easy.
- This model gives results which are often in very good agreement with experiments.

A consequence is that the damping coefficient ξ can only be determined by experiments.

MULTIPLE DEGREES OF FREEDOM

SDF $m \ddot{u} + c \dot{u} + k u = p(t)$

MDF $[\mathbf{m}]\{\ddot{\mathbf{u}}\} + [\mathbf{c}]\{\dot{\mathbf{u}}\} + [\mathbf{k}]\{\mathbf{u}\} = \{\mathbf{p}(t)\}$

$[\mathbf{k}]$ stiffness matrix

$[\mathbf{c}]$ damping matrix

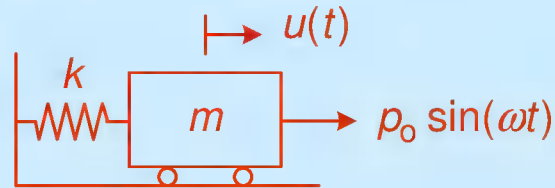
$[\mathbf{m}]$ mass matrix

The mass and stiffness matrices are obtained by (finite element) discretisation of the structure.

The damping matrix cannot be obtained by discretisation, a different approach must be used.

SDF - HARMONIC EXCITATIONS

A harmonic load is applied to the structure.

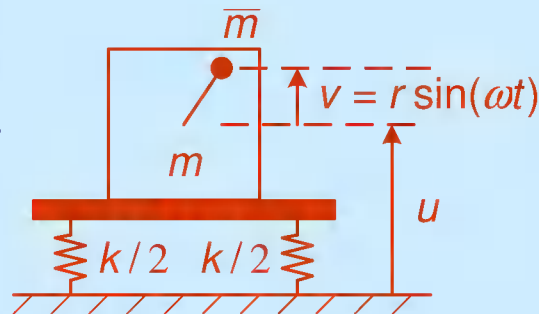


Without damping, the equation of motion is

$$m \ddot{u} + k u = p_0 \sin(\omega t)$$

Example 1

The system consisting of the mass m and the excentric mass $\bar{m}$ is considered for writing Newton's equation.



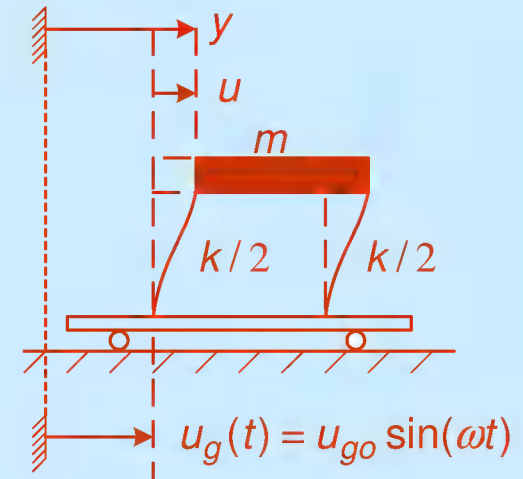
$$(m - \bar{m}) \ddot{u} + \bar{m} (\ddot{v} + \ddot{u}) = -k u$$

$$m \ddot{u} + k u = \bar{m} r \omega^2 \sin(\omega t)$$

Example 2

$$y = u + u_g$$

$$m \ddot{y} = -k u$$



If u is to be studied (e.g. earthquake)

$$m (\ddot{u} + \ddot{u}_g) = -k u$$

$$\rightarrow m \ddot{u} + k u = m u_{g0} \omega^2 \sin(\omega t)$$

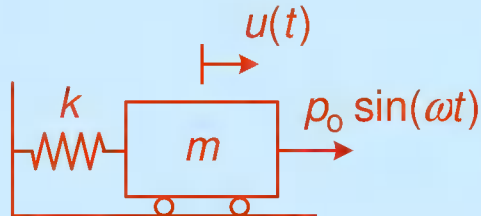
If y is to be studied (e.g. floor isolation)

$$m \ddot{y} = -k (y - u_g)$$

$$\rightarrow m \ddot{y} + k y = k u_{g0} \sin(\omega t)$$

WITHOUT DAMPING

$$m \ddot{u} + k u = p_o \sin(\omega t)$$



The solution $u(t)$ of the differential equation is the sum of two parts $u_h(t)$ and $u_p(t)$.

$$u(t) = u_h(t) + u_p(t)$$

homogeneous solution $u_h(t) = C \sin(\omega_n t + \theta)$

particular solution $u_p(t) = A \sin(\omega t)$

$$\ddot{u}_p(t) = -A \omega^2 \sin(\omega t)$$

$$\rightarrow -m A \omega^2 + k A = p_o \rightarrow A = \frac{p_o}{k - m \omega^2} = \frac{p_o / k}{1 - (\omega / \omega_n)^2}$$

$$u(t) = C \sin(\omega_n t + \theta) + \frac{p_o / k}{1 - (\omega / \omega_n)^2} \sin(\omega t)$$

C and θ are determined by the initial conditions

$u(t)$ is a summation of two sinus and is not defined for $\omega = \omega_n$.

In reality, the damping implies that $u_h(t)$ disappears after some time. Then the solution (*steady state response*) is

$$u(t) = u_p(t) = \frac{p_o / k}{1 - (\omega / \omega_n)^2} \sin(\omega t)$$

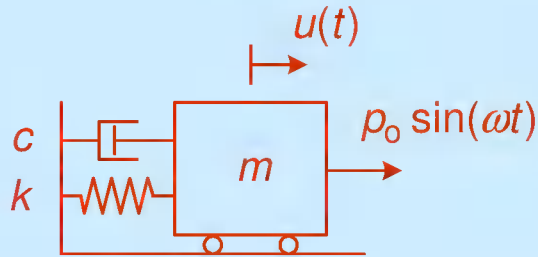
conclusions

- After some time, the structure vibrates with the same frequency as the applied force.
- The amplitudes of the vibration are infinite when $\omega = \omega_n$.

$u_h(t)$ is the transient response

$u_p(t)$ is the steady state response

WITH VISCOUS DAMPING



$$m \ddot{u} + c \dot{u} + k u = p_0 \sin(\omega t)$$

The homogeneous solution $u_h(t)$ (transient response) disappears after some time.

$$u_h(t) = C e^{-\xi \omega_n t} \sin(\omega_d t + \theta)$$

The particular solution $u_p(t)$ (steady state response) is of the form:

$$u_p(t) = A \sin(\omega t - \phi)$$

after calculations (see the book), it is obtained

$$u_p(t) = \frac{p_0/k}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\xi(\omega/\omega_n)]^2}} \sin(\omega t - \phi)$$

$$\tan \phi = \frac{2\xi(\omega/\omega_n)}{1 - (\omega/\omega_n)^2} \quad 0 < \phi < 180^\circ$$

Remarks

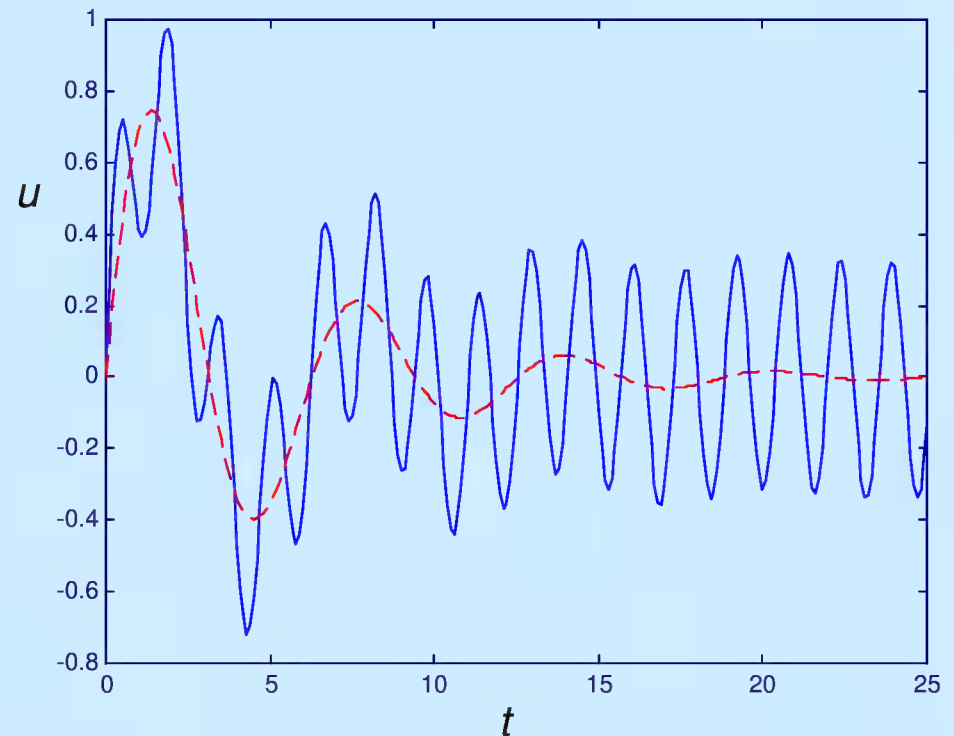
- The total response is $u(t) = u_h(t) + u_p(t)$. But after some time $u_h(t)$ disappears and $u(t) = u_p(t)$ (steady state response).
- After some time, the structure vibrates with the same frequency as the applied force.

This numerical example shows that the transient response $u_h(t)$ disappears after some time and that only the steady state response $u_p(t)$ is then left.

$$u(t) = u_h(t) + u_p(t)$$

$$u(t) = C e^{-\xi \omega_n t} \sin(\omega_d t + \theta) + \frac{p_0/k}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\xi(\omega/\omega_n)]^2}} \sin(\omega t - \phi)$$

C and θ are determined by the initial conditions.



— total response $u(t)$
 - - - transient response $u_h(t)$

For this case, the steady state is obtained after about 20 seconds.

Dynamic factor

After some while, the structure vibrates with the same frequency as the applied force. It is the steady state response $u_p(t)$.

The amplitude of these vibrations are now studied.

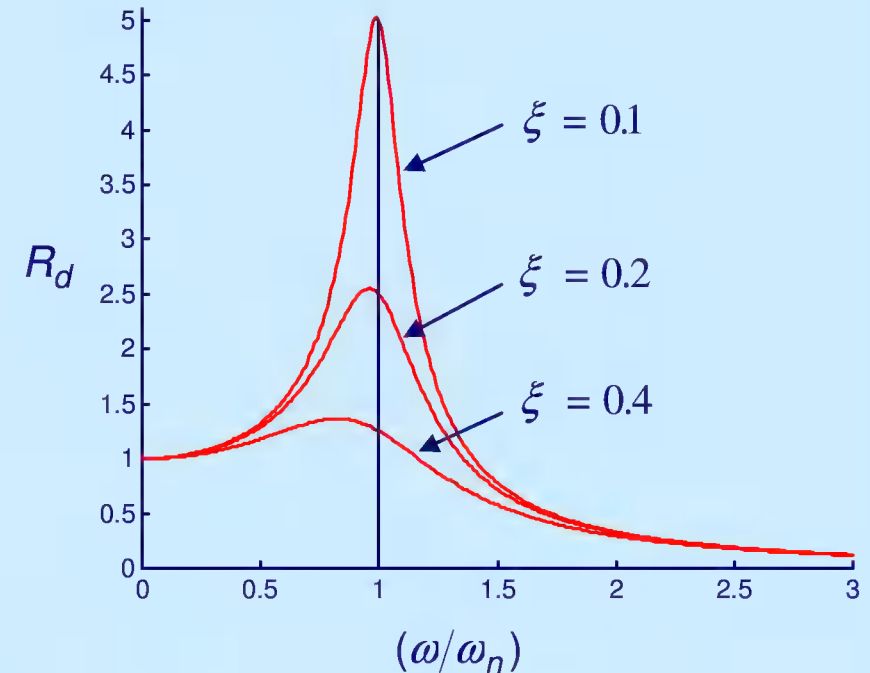
$$u_p(t) = \frac{p_o/k}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\xi(\omega/\omega_n)]^2}} \sin(\omega t - \phi)$$

The static deformation due to a static load p_o is $(u_{st})_o = \frac{p_o}{k}$

The amplitude of the vibration is equal to the product of the static deformation times a dimensionless dynamic factor R_d .

$$R_d(\xi, \omega/\omega_n) = \frac{1}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\xi(\omega/\omega_n)]^2}}$$

R_d can be plotted as function of the ratio ω/ω_n for different values of the damping coefficient ξ .



$\omega/\omega_n < 0.25 \rightarrow R_d \approx 1$ "quasi static" response

~~$$m\ddot{u} + c\dot{u} + ku = p_o \sin(\omega t) \rightarrow u = p_o/k \sin(\omega t)$$~~

$\omega \rightarrow \omega_n$ the amplitudes of vibrations become large : **Resonance**

Resonance

Resonance is reached for

$$\omega = \omega_r = \omega_n \sqrt{1 - 2\xi^2}$$

For this value of ω , the dynamic factor is

$$R_{d\max} = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

if $\xi < 0.1$ then $R_{d\max} \approx \frac{1}{2\xi}$ and

$$\omega_n = \sqrt{k/m} \approx \omega_D = \omega_n \sqrt{1 - \xi^2} \approx \omega_r = \omega_n \sqrt{1 - 2\xi^2}$$

Example

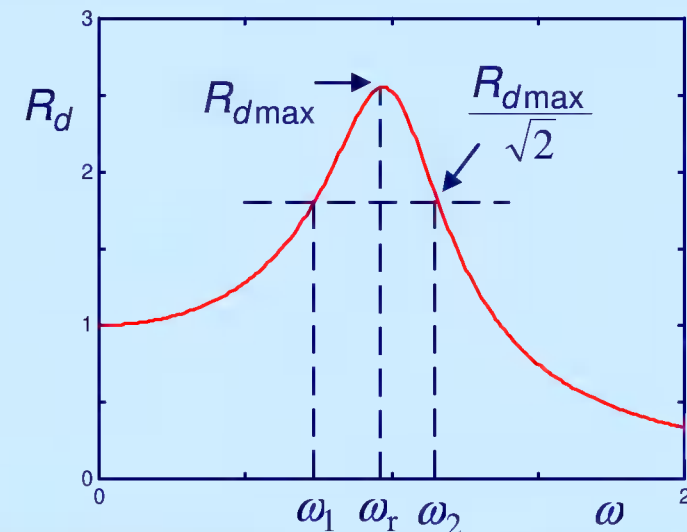
$$\xi = 2\% \rightarrow R_d = 25$$

The deformations are 25 times the static ones.

Band-width method

(Experimental method to determine ξ)

The structure is excited by a harmonic load. The frequency of the load is increased step by step. At each step, the amplitudes of vibrations of the steady state response are measured. This implies that at each step, some time must be waited so that the transient response disappears. The curve R_d as function of ω is then obtained experimentally.

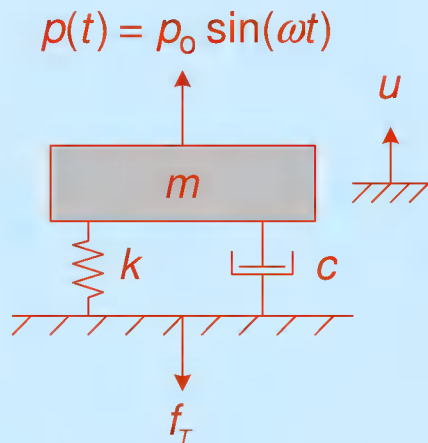


$$\xi < 0.1 \rightarrow \xi \approx \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1} = \frac{f_2 - f_1}{f_2 + f_1}$$

$$f = \frac{\omega}{2\pi}$$

Force transmission and vibration isolation

A harmonic load is applied to a structure. This structure is connected to the ground through a support modelled by a spring k and a damper c .



Steady state response

$$u(t) = \frac{p_0}{k} R_d \sin(\omega t - \phi)$$

The force transmitted to the ground is

$$f_T(t) = ku(t) + c\dot{u}(t)$$

$$= p_0 R_d \sin(\omega t - \phi) + \frac{p_0 c}{k} R_d \omega \cos(\omega t - \phi)$$

$$\begin{aligned} f_T(t) &= p_0 R_d [\sin(\omega t - \phi) + 2\xi(\omega/\omega_n) \cos(\omega t - \phi)] \\ &= p_0 R_d \sqrt{1 + [2\xi(\omega/\omega_n)]^2} \sin(\omega t - \phi + \alpha) \end{aligned}$$

The transmissibility TR is defined as the ratio between the amplitude of the transmitted force f_T and the amplitude of force applied to the structure.

$$TR = \frac{f_{T\max}}{p_0} = \sqrt{\frac{1 + [2\xi(\omega/\omega_n)]^2}{[1 - (\omega/\omega_n)^2]^2 + [2\xi(\omega/\omega_n)]^2}}$$

TR is dimensionless.

The objective is to choose the support (k , c) such that TR is as small as possible.

The mass m of the structure and the frequency of the load ω are imposed. The problem is to choose a support with k and ξ such that the transmissibility TR is minimal.

TR as function of the ratio ω / ω_n is plotted for different values of ξ .

$$\omega_n = \sqrt{k/m}$$

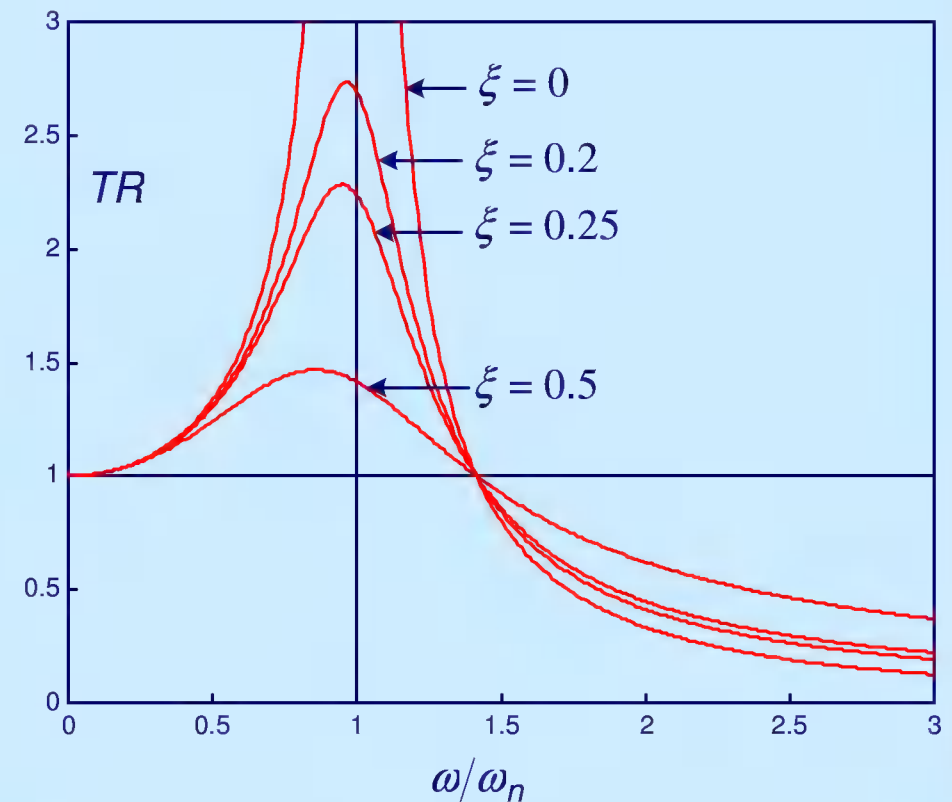
The transmitted force is less than the applied one if the natural frequency ω_n is such that $\omega / \omega_n > 1.4$.

A low TR is obtained for low values of ω_n and ξ .

However, a very low ω_n implies a low k and therefore a too large static displacement p_0 / k .

Besides, a very low ξ implies high displacement amplitude while passing through resonance which may occur before the load reaches the circular frequency ω .

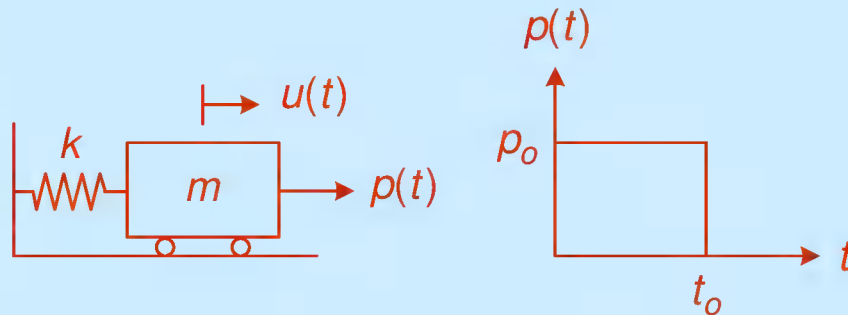
Natural rubber is a good compromise and is often used for the isolation of vibrations.



SDF - ARBITRARY EXCITATIONS

This chapter studies the response of a SDF system to pulse, impulse and periodic excitations.

RECTANGULAR PULSE FORCE



initial conditions: $u_o = \dot{u}_o = 0$

An undamped SDF system is loaded by a rectangular pulse force. For this case, the differential equation can be solved analytically.

$$0 \leq t \leq t_o$$

$$m \ddot{u} + k u = p_o$$

$$u(t) = u_h(t) + u_p(t)$$

$$u_h(t) = A \cos(\omega_n t) + B \sin(\omega_n t)$$

$$u_p(t) = \frac{p_o}{k}$$

$$u(t) = A \cos(\omega_n t) + B \sin(\omega_n t) + \frac{p_o}{k}$$

$$\begin{cases} u_o = 0 \\ \dot{u}_o = 0 \end{cases} \rightarrow \begin{cases} A + \frac{p_o}{k} = 0 \\ B \omega_n = 0 \end{cases}$$

$$u(t) = \frac{p_o}{k} [1 - \cos(\omega_n t)]$$

$$t \geq t_o \quad m \ddot{u} + k u = 0 \quad \text{free vibration}$$

$$u(t) = u(t_o) \cos(\omega_n t_1) + \frac{\dot{u}(t_o)}{\omega_n} \sin(\omega_n t_1)$$

$$t_1 = t - t_o \quad u(t_o) = \frac{p_o}{k} [1 - \cos(\omega_n t_o)]$$

$$\dot{u}(t_o) = \frac{p_o \omega_n}{k} \sin(\omega_n t_o)$$

$$u(t) = \frac{p_o}{k} [1 - \cos(\omega_n t_o)] \cos(\omega_n t - \omega_n t_o) \\ + \frac{p_o}{k} \sin(\omega_n t_o) \sin(\omega_n t - \omega_n t_o)$$

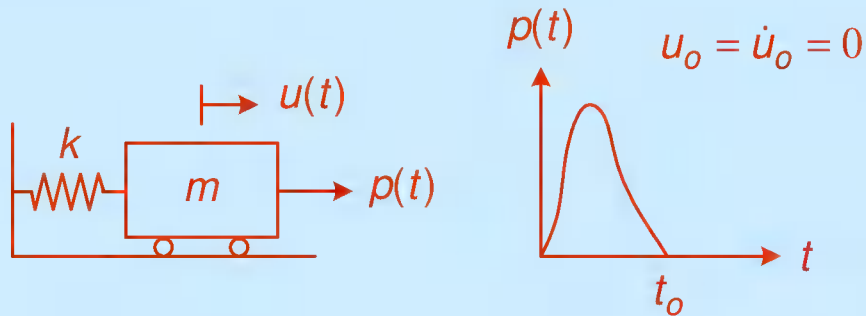
$$u(t) = \frac{p_o}{k} [\cos(\omega_n t - \omega_n t_o) - \cos(\omega_n t_o) \cos(\omega_n t - \omega_n t_o) \\ + \sin(\omega_n t_o) \sin(\omega_n t - \omega_n t_o)]$$

by using $\cos a \cos b - \sin a \sin b = \cos(a + b)$

it is obtained

$$u(t) = \frac{p_o}{k} [\cos(\omega_n t - \omega_n t_o) - \cos(\omega_n t)]$$

IMPULSE LOAD



An undamped SDF system is loaded by a short impulse force. An approximative response is to be calculated.

Newton's law for $t \leq t_o$
$$m \frac{d\dot{u}(t)}{dt} + k u(t) = p(t)$$

Hypothesis: t_o is so small that the displacement u is still zero at t_o . ($u_o = 0$).

$$\begin{cases} t_o \text{ is very small} \\ u_o = 0 \end{cases} \rightarrow u(t_o) \approx 0$$

Then $ku(t)$ can be neglected in Newton's equation

$$d\dot{u}(t) = \frac{1}{m} p(t) dt \rightarrow \dot{u}(t_o) - \dot{u}_o = \frac{1}{m} \int_0^{t_o} p(t) dt$$

$$\rightarrow \dot{u}(t_o) = \frac{I}{m} \quad I = \int_0^{t_o} p(t) dt$$

At t_o , the impulse load is assumed to have produced an initial velocity, but no displacement.

Newton's law for $t \geq t_o$ (free vibration)

$$u(t) = u(t_o) \cos[\omega_n(t - t_o)] + \frac{\dot{u}(t_o)}{\omega_n} \sin[\omega_n(t - t_o)]$$

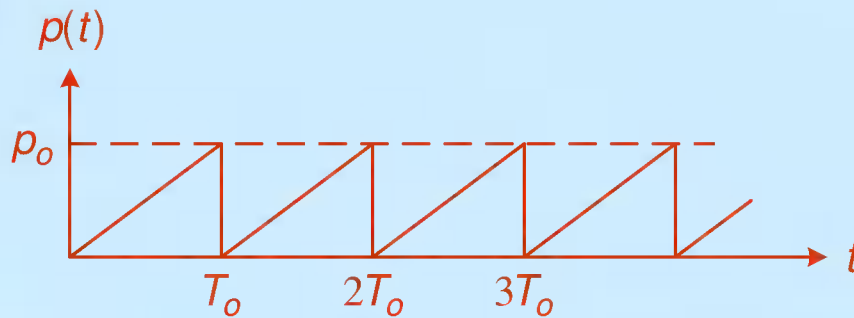
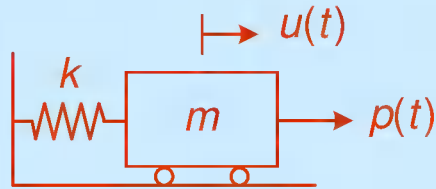
→ the maximum displacement is
$$u_{\max} = \frac{I}{m\omega_n}$$

This result is only valid if t_o is small enough so that $u(t_o) \approx 0$. In practice it means $t_o < T_n/10$

In such case, the maximal deformation does not depend on the form of the impulse load, but only on the value of the integral I .

PERIODIC EXCITATION

A SDF system is excited by a periodic (but not harmonic) load.



Idea : a periodic function can be separated into its harmonic components using *Fourier series*.

$$p(t) = a_o + \sum_{j=1}^{\infty} [a_j \cos(j\omega_o t) + b_j \sin(j\omega_o t)]$$

$$\omega_o = \frac{2\pi}{T_o}$$

$$a_j = \frac{2}{T_o} \int_0^{T_o} p(t) \cos(j\omega_o t) dt$$

$$a_o = \frac{1}{T_o} \int_0^{T_o} p(t) dt$$

$$b_j = \frac{2}{T_o} \int_0^{T_o} p(t) \sin(j\omega_o t) dt$$

Newton's equation is

$$m\ddot{u} + ku = p(t) = a_o + \sum_{j=1}^{\infty} [a_j \cos(j\omega_o t) + b_j \sin(j\omega_o t)]$$

The **steady state response** is calculated by using the theorem of superposition.

$$m\ddot{u} + ku = a_o \rightarrow u_p = \frac{a_o}{k}$$

$$m\ddot{u} + ku = b_j \sin(j\omega_o t) \rightarrow u_p = \frac{b_j/k}{1 - (j\omega_o / \omega_n)^2} \sin(j\omega_o t)$$

$$m\ddot{u} + ku = b_j \sin(j\omega_o t) \rightarrow u_p = \frac{b_j/k}{1 - (j\omega_o / \omega_n)^2} \sin(j\omega_o t)$$

The total **steady state response** is then

$$u(t) = \frac{a_o}{k} + \sum_{j=1}^{\infty} \frac{1/k}{1 - (j\omega_o / \omega_n)^2} [a_j \cos(j\omega_o t) + b_j \sin(j\omega_o t)]$$

Example $0 \leq t \leq T_o$ $p(t) = \frac{p_o}{T_o} t$

$$a_o = \frac{1}{T_o} \int_0^{T_o} \frac{p_o}{T_o} t dt = \frac{p_o}{2}$$

$$a_j = \frac{2}{T_o} \int_0^{T_o} \frac{p_o}{T_o} t \cos(j\omega_o t) dt = 0$$

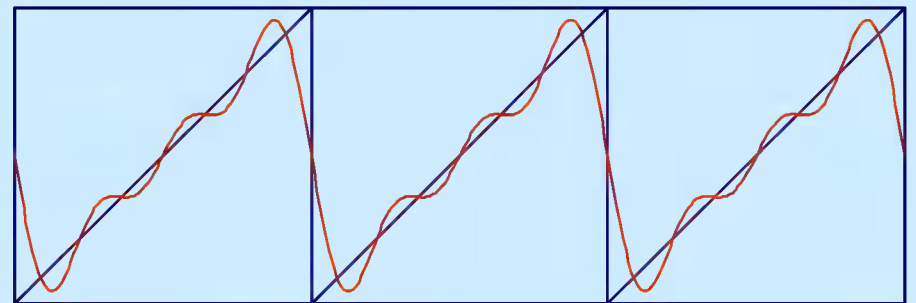
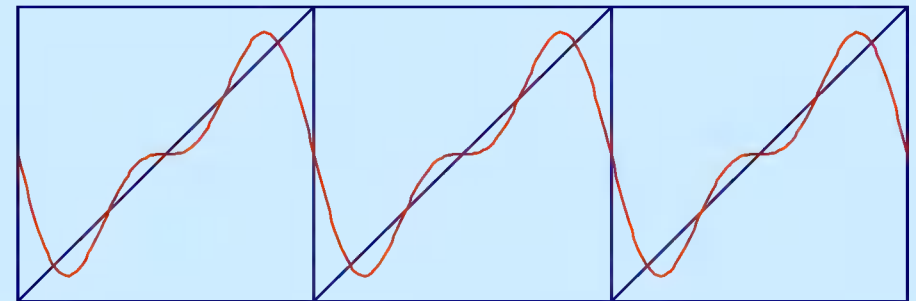
$$b_j = \frac{2}{T_o} \int_0^{T_o} \frac{p_o}{T_o} t \sin(j\omega_o t) dt = -\frac{p_o}{j\pi}$$

The total **steady state response** is then

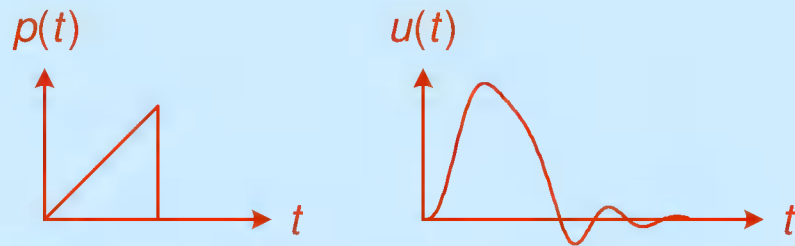
$$u(t) = \frac{p_o}{2k} - \sum_{j=1}^{\infty} \frac{p_o / j\pi k}{1 - (j\omega_o / \omega_n)^2} \sin(j\omega_o t)$$

with $\frac{\omega_o}{\omega_n} = 2$ $u(t) = \frac{p_o}{2k} - \frac{p_o}{\pi k} \left[\frac{\sin(\omega_o t)}{-3} + \frac{\sin(2\omega_o t)}{-30} \right. \\ \left. + \frac{\sin(3\omega_o t)}{-105} + \frac{\sin(4\omega_o t)}{-252} \dots \right]$

Only the first three Fourier terms in $u(t)$ must be considered to get an error less than 2 %. The plot of the load approximations with 2 and 3 Fourier terms shows that an inaccurate approximation of the load gives an accurate approximation of the response. The reason is that the higher frequencies in the load do not give any contributions to the response.



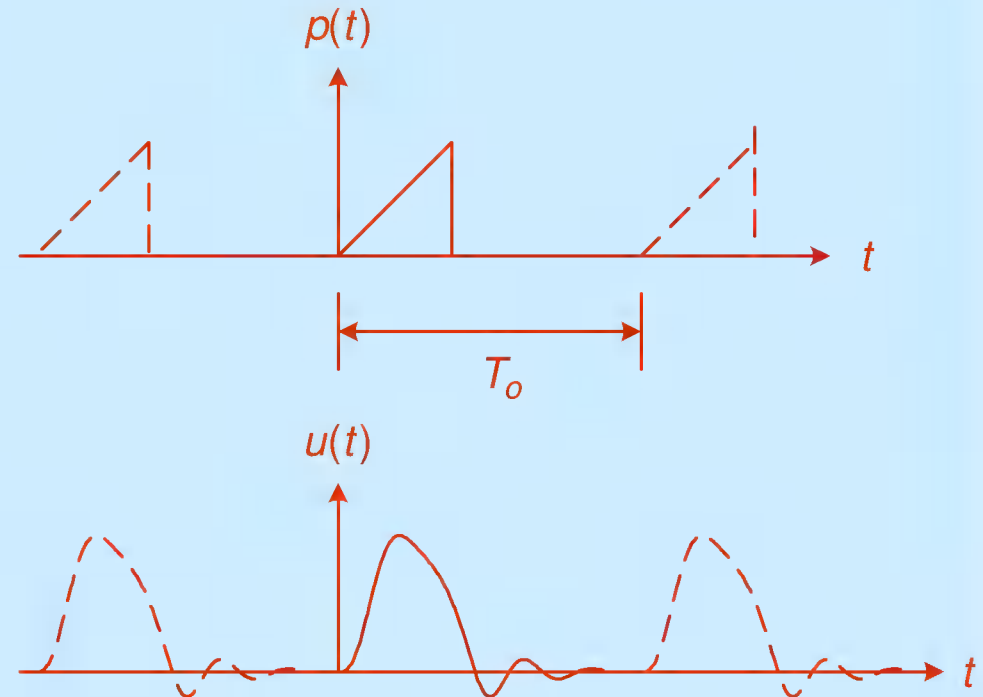
NON PERIODIC EXCITATION



Fourier series can also be used for non periodic loads. As example, the response $u(t)$ of a damped SDF system loaded by the force $p(t)$ is given above.

$p(t)$ and therefore $u(t)$ are non periodic. They become periodic by artificially adding the same model. Then a Fourier analysis can be done.

The Fourier analysis of the artificial periodic problem is performed by only considering the steady state response. But the result obtained for the real problem includes both the transient and steady state response.



Remark: $u(t)$ must be zero at the end of the period T_o .

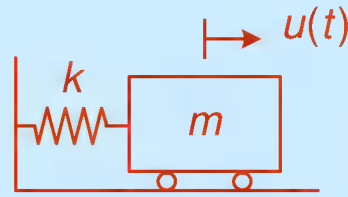
Such an approach is not used for real problems, but it constitutes the basis for signal analysis, see the chapter experimental dynamics.

RAYLEIGH'S METHOD

Rayleigh's method can be used to calculate approximately the lowest natural frequency of beams.

INTRODUCTION

Free vibrations of an undamped SDF system are considered.



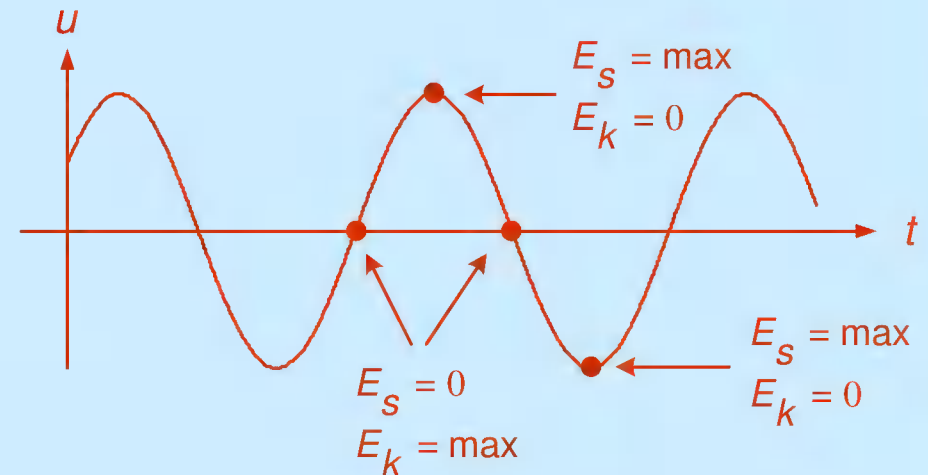
$$u(t) = u_o \sin(\omega_n t) \quad \dot{u}(t) = u_o \omega_n \cos(\omega_n t)$$

strain energy

$$E_s(t) = \frac{1}{2} k u(t)^2 = \frac{1}{2} k u_o^2 \sin^2(\omega_n t)$$

kinetic energy

$$E_k(t) = \frac{1}{2} m \dot{u}(t)^2 = \frac{1}{2} m \omega_n^2 u_o^2 \cos^2(\omega_n t)$$



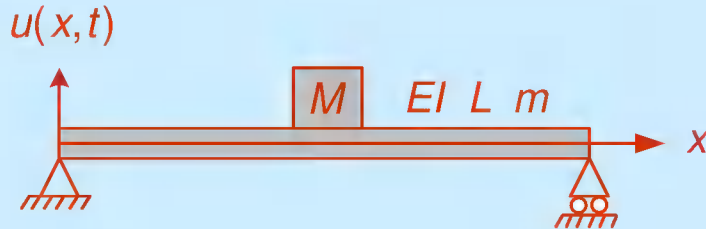
$$E_{s\max} = \frac{1}{2} k u_o^2 \quad E_{k\max} = \frac{1}{2} m \omega_n^2 u_o^2$$

conservation of energy

$$E_{s\max} = E_{k\max} \rightarrow \frac{1}{2} k u_o^2 = \frac{1}{2} m \omega_n^2 u_o^2$$

$$\rightarrow \boxed{\omega_n = \sqrt{\frac{k}{m}}}$$

APPLICATION



The same approach can be used to calculate an approximative value of the lowest natural frequency of the beam above.

The method consists in

- estimating the vibration shape (eigenmode)
- calculate $E_s(t)$ and $E_k(t)$
- using $E_s(t) = E_k(t)$ to get the natural frequency

Let's take $u(x, t) = Y \sin\left(\frac{\pi x}{L}\right) \sin(\omega t)$

which is the exact eigenmode if $M = 0$.

strain energy

$$u''(x, t) = \frac{\partial^2 u(x, t)}{\partial x^2} = -\frac{\pi^2}{L^2} Y \sin\left(\frac{\pi x}{L}\right) \sin(\omega t)$$

$$\begin{aligned} E_s &= E_{s\text{beam}} = \int_0^L \frac{1}{2} EI [u''(x, t)]^2 dx \\ &= \frac{1}{2} EI Y^2 \frac{\pi^4}{L^4} \sin^2(\omega t) \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx \end{aligned}$$

$$\int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{1}{2} \int_0^L \left[1 - \cos\left(\frac{2\pi x}{L}\right)\right] dx = \frac{L}{2}$$

$$\cos(2a) = 1 - 2\sin^2(a)$$

$$E_{s\text{max}} = \frac{1}{2} EI Y^2 \frac{\pi^4}{2L^3}$$

kinetic energy

$$\dot{u}(x,t) = \frac{\partial u(x,t)}{\partial t} = Y \omega \sin\left(\frac{\pi x}{L}\right) \cos(\omega t)$$

$$E_k = E_{kM} + E_{k\text{beam}}$$

$$E_{kM} = \frac{1}{2} M [\dot{u}(L/2, t)]^2 = \frac{1}{2} M Y^2 \omega^2 \cos^2(\omega t)$$

$$\begin{aligned} E_{k\text{beam}} &= \int_0^L \frac{1}{2} m [\dot{u}(x, t)]^2 dx \\ &= \frac{1}{2} m Y^2 \omega^2 \cos^2(\omega t) \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx \end{aligned}$$

$$E_{k\max} = \frac{1}{2} Y^2 \omega^2 \left(M + \frac{mL}{2} \right)$$

conservation of energy

$$E_{k\max} = E_{s\max} \rightarrow \omega^2 \left(M + m \frac{L}{2} \right) = EI \frac{\pi^4}{2L^3}$$

$$\rightarrow \omega = \pi^2 \sqrt{\frac{EI}{L^3(2M + mL)}}$$

particular cases

$$M = 0 \quad \omega = \pi^2 \sqrt{\frac{EI}{mL^4}} \quad \text{exact solution since the eigenmode is exact.}$$

$$m = 0 \quad \omega = \frac{\pi^2}{\sqrt{2}} \sqrt{\frac{EI}{ML^3}} = 6.98 \sqrt{\frac{EI}{ML^3}}$$

In that case, the beam is considered as a spring with $k = \frac{48EI}{L^3}$

which gives as exact solution $\omega = \sqrt{\frac{k}{M}} = \sqrt{\frac{48EI}{ML^3}} = 6.93 \sqrt{\frac{EI}{ML^3}}$

REMARKS

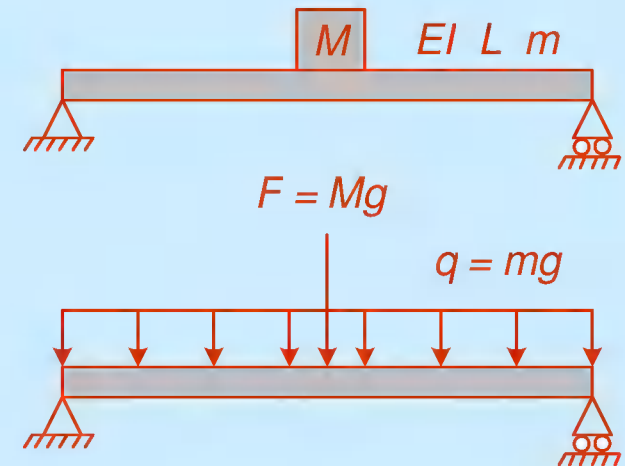
- Only an estimate of the natural frequency can be calculated.
- The accuracy of the result depends entirely on the shape function which is assumed to represent the eigenmode.
- The natural frequency calculated by Rayleigh's method is always greater than the exact value.

SELECTION OF THE SHAPE FUNCTION

The shape function (eigenmode) must be kinematically admissible, i.e. must satisfy the displacements boundary conditions at the supports.

a possibility is to take the deflected shape corresponding to the weight of the structure. (implemented in some f.e.m. codes)

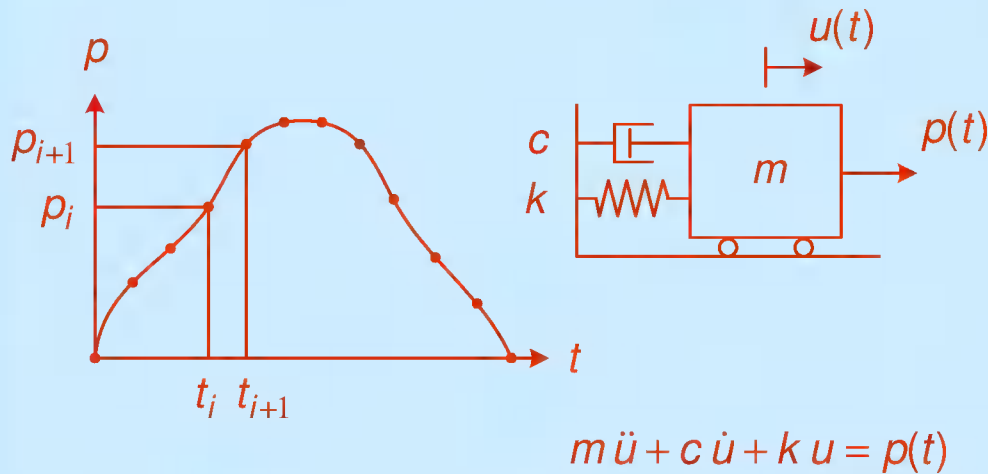
example



CONCLUSION

The main interest of the Rayleigh's method lies in its ability to provide useful estimation of the natural frequency from any reasonable assumption of the eigenmode.

SDF - TIME INTEGRATION METHODS



The load p is time discretised.

$$\mathbf{p} = [p_0 \ p_1 \ p_2 \ \dots \ p_{i-1} \ p_i \ p_{i+1} \ \dots \ p_n]$$

The purpose is to calculate u (and $\dot{u}$, $\ddot{u}$ if required) at the discrete time instants.

$$\mathbf{u} = [u_0 \ u_1 \ u_2 \ \dots \ u_{i-1} \ u_i \ u_{i+1} \ \dots \ u_n]$$

$$\dot{\mathbf{u}} = [\dot{u}_0 \ \dot{u}_1 \ \dot{u}_2 \ \dots \ \dot{u}_{i-1} \ \dot{u}_i \ \dot{u}_{i+1} \ \dots \ \dot{u}_n]$$

$$\ddot{\mathbf{u}} = [\ddot{u}_0 \ \ddot{u}_1 \ \ddot{u}_2 \ \dots \ \ddot{u}_{i-1} \ \ddot{u}_i \ \ddot{u}_{i+1} \ \dots \ \ddot{u}_n]$$

The response at time step $i+1$ is calculated from the equation of motion, a difference expression, and known responses at one or more preceding time steps.

Equations of motion at time i and $i+1$.

$$m\ddot{u}_i + c\dot{u}_i + ku_i = p_i \quad (1)$$

$$m\ddot{u}_{i+1} + c\dot{u}_{i+1} + ku_{i+1} = p_{i+1} \quad (2)$$

An **explicit algorithm** uses a difference expression of the general form

$$u_{i+1} = f(u_i, \dot{u}_i, \ddot{u}_i, u_{i-1}, \dot{u}_{i-1}, \dots)$$

which is combined with equation (1)

An **implicit algorithm** uses a difference expression of the general form

$$u_{i+1} = f(\dot{u}_{i+1}, \ddot{u}_{i+1}, u_i, \dot{u}_i, \ddot{u}_i, u_{i-1}, \dots)$$

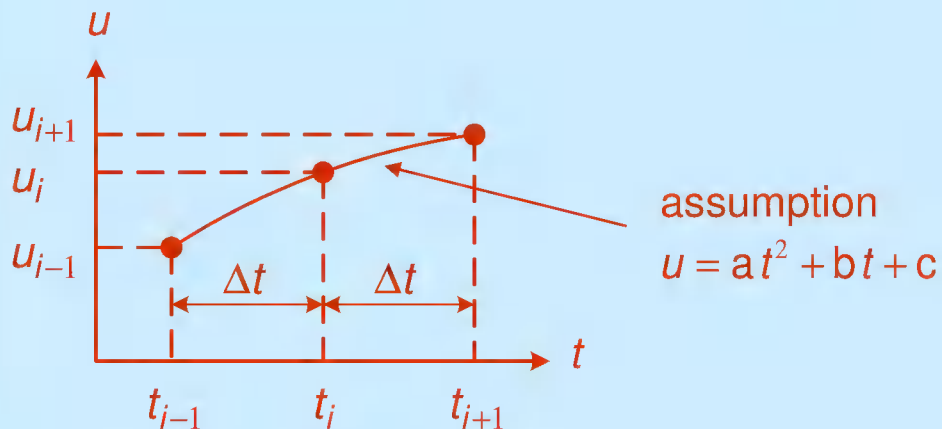
which is combined with equation (2)

CENTRAL DIFFERENCE METHOD

(explicit algorithm)

difference expressions :

$$\dot{u}_i = \frac{u_{i+1} - u_{i-1}}{2\Delta t} \quad \ddot{u}_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2}$$



The displacement curve is assumed to be a parabola between 3 consecutive points.

Introducing the two difference expressions in the equilibrium equation at time i

$$m \ddot{u}_i + c \dot{u}_i + k u_i = p_i$$

gives the unknown u_{i+1} as

$$\hat{k} u_{i+1} = \hat{p}_i \quad \hat{k} = \frac{m}{(\Delta t)^2} + \frac{c}{2\Delta t}$$

$$\hat{p}_i = p_i - \left[\frac{m}{(\Delta t)^2} - \frac{c}{2\Delta t} \right] u_{i-1} - \left[k - \frac{2m}{(\Delta t)^2} \right] u_i$$

initialisation

The given initial conditions are u_0 and $\dot{u}_0$.

But u_0 and u_{-1} are required to determine u_1 .

The difference expressions for $i = 0$ are

$$\dot{u}_0 = \frac{u_1 - u_{-1}}{2\Delta t} \quad \ddot{u}_0 = \frac{u_1 - 2u_0 + u_{-1}}{(\Delta t)^2}$$

which after substitution gives

$$u_{-1} = u_0 - \Delta t \dot{u}_0 + \frac{(\Delta t)^2}{2} \ddot{u}_0$$

The equation of motion at time $t = 0$ gives the initial acceleration as

$$\ddot{u}_0 = \frac{p_0 - c \dot{u}_0 - k u_0}{m}$$

Algorithm – central difference method

1 initial calculations

$$\ddot{u}_0 = \frac{p_0 - c \dot{u}_0 - k u_0}{m} \quad u_{-1} = u_0 - \Delta t \dot{u}_0 + \frac{(\Delta t)^2}{2} \ddot{u}_0$$

$$\hat{k} = \frac{m}{(\Delta t)^2} + \frac{c}{2\Delta t} \quad a = \frac{m}{(\Delta t)^2} - \frac{c}{2\Delta t} \quad b = k - \frac{2m}{(\Delta t)^2}$$

2 calculations for step i

$$\hat{p}_i = p_i - a u_{i-1} - b u_i \quad u_{i+1} = \hat{p}_i / \hat{k}$$

$$\text{if required } \dot{u}_i = \frac{u_{i+1} - u_{i-1}}{2\Delta t} \quad \ddot{u}_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta t)^2}$$

3 repetition for the next time step

replace i by $i + 1$ and repeat step 2

AVERAGE ACCELERATION METHOD

(implicit algorithm)

difference expressions :

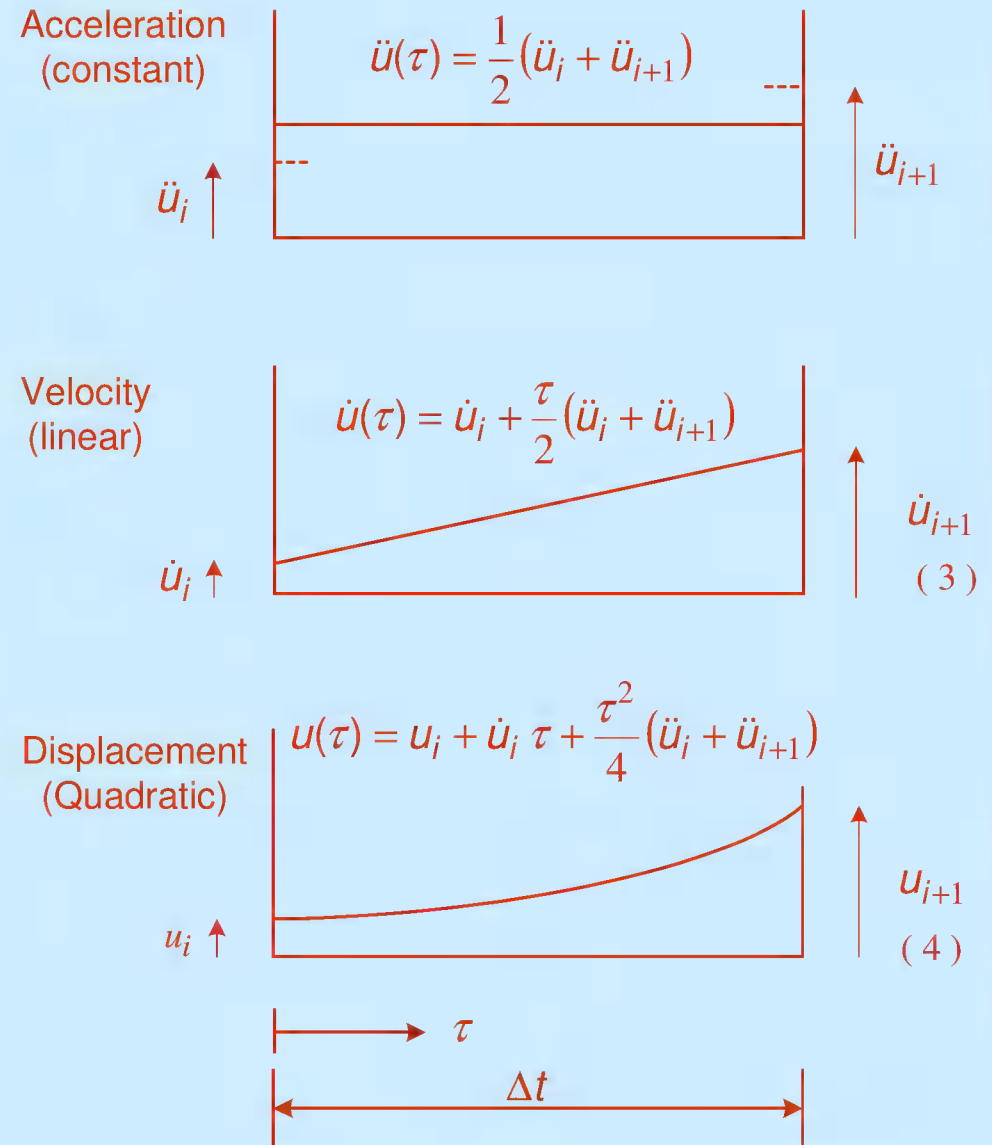
$$\dot{u}_{i+1} = \dot{u}_i + \frac{\Delta t}{2}(\ddot{u}_i + \ddot{u}_{i+1}) \quad (3)$$

$$u_{i+1} = u_i + \dot{u}_i \Delta t + \frac{(\Delta t)^2}{4}(\ddot{u}_i + \ddot{u}_{i+1}) \quad (4)$$

The acceleration is assumed constant between t_i and t_{i+1} . it is given by

$$\ddot{u}(\tau) = \frac{1}{2}(\ddot{u}_i + \ddot{u}_{i+1}) \quad 0 \leq \tau \leq \Delta t$$

The velocity is then linear and the displacement quadratic, which gives equations (3) and (4).



The system constituted by equations (2) (3) (4) has 3 unknowns, $\ddot{u}_{i+1}$, $\dot{u}_{i+1}$ and u_{i+1} .

The solution of the system can be calculated as

$$\hat{k} u_{i+1} = \hat{p}_i \quad \hat{k} = k + \frac{2c}{\Delta t} + \frac{4m}{(\Delta t)^2}$$

$$\hat{p} = p_{i+1} + c \left[\frac{2u_i}{\Delta t} + \dot{u}_i \right] + m \left[\frac{4u_i}{(\Delta t)^2} + \frac{4\dot{u}_i}{\Delta t} + \ddot{u}_i \right]$$

$$\dot{u}_{i+1} = \frac{2}{\Delta t} (u_{i+1} - u_i) - \dot{u}_i$$

$$\ddot{u}_{i+1} = \frac{1}{m} (p_{i+1} - c \dot{u}_{i+1} - k u_{i+1})$$

The given initial conditions are u_0 and $\dot{u}_0$.

$\ddot{u}_0$ can be calculated by $\ddot{u}_0 = \frac{p_0 - c \dot{u}_0 - k u_0}{m}$

Algorithm – average acceleration method

1 initial calculations

$$\ddot{u}_0 = \frac{p_0 - c \dot{u}_0 - k u_0}{m} \quad \hat{k} = k + \frac{2c}{\Delta t} + \frac{4m}{(\Delta t)^2}$$

2 calculations for step i

$$\hat{p} = p_{i+1} + c \left[\frac{2u_i}{\Delta t} + \dot{u}_i \right] + m \left[\frac{4u_i}{(\Delta t)^2} + \frac{4\dot{u}_i}{\Delta t} + \ddot{u}_i \right]$$

$$\hat{k} u_{i+1} = \hat{p}_i \quad \dot{u}_{i+1} = \frac{2}{\Delta t} (u_{i+1} - u_i) - \dot{u}_i$$

$$\ddot{u}_{i+1} = \frac{1}{m} (p_{i+1} - c \dot{u}_{i+1} - k u_{i+1})$$

3 repetition for the next time step

replace i by $i + 1$ and repeat step 2

STABILITY

The central difference method will blow up, giving meaningless results, if the time step chosen is not short enough. The specific requirements for stability is :

$$\frac{\Delta t}{T_n} < \frac{1}{\pi}$$

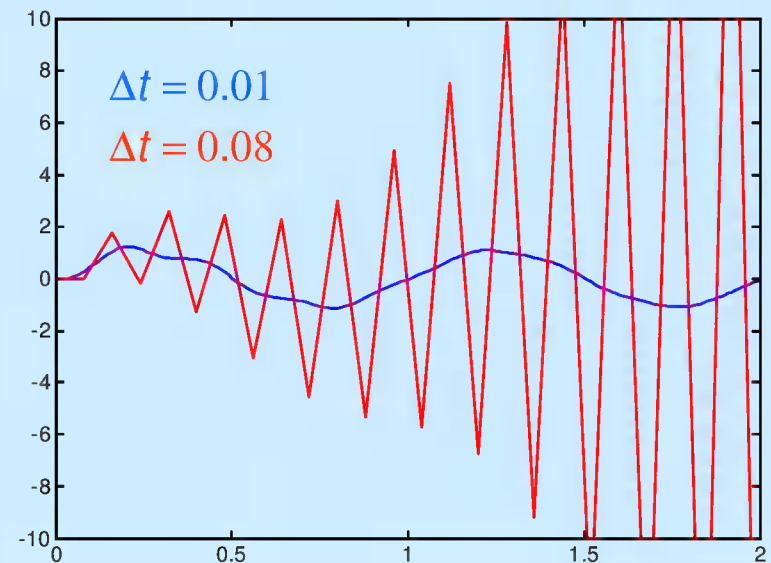
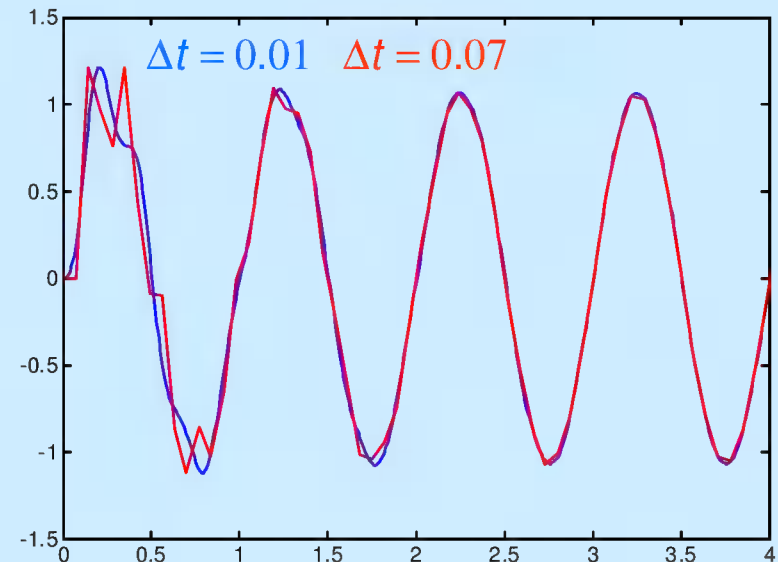
The average acceleration method is unconditionally stable, which means that the procedure leads to bounded solutions regardless of the time step length.

example

$$m \ddot{u} + c \dot{u} + k u = \sin(2\pi t)$$

$$\omega_n = 8\pi \quad k = 1 \quad \xi = 0.05$$

central difference method : $\Delta t < 0.0796$



HOW TO CHOOSE Δt ?

Δt must be small enough to get a good accuracy, and long enough to be computationally efficient.

For SDF systems, stability criteria are not restrictive since Δt must be considerably smaller than the stability limit to ensure adequate accuracy.

Stability of the numerical method is however important for the analysis of multiple degrees of freedom systems where it is often necessary to use unconditionally stable methods.

One useful technique for selecting the time step is to solve the problem with a time step that seems reasonable, then repeat the solution with a smaller time step and compare the results, continuing the process until to successive solutions are close enough.

NONLINEAR RESPONSE

If the structure undergoes large displacements or plastic deformation, the equation becomes

$$m \ddot{u} + c \dot{u} + f(u) = p(t)$$

where the internal force $f(u)$ is a nonlinear function.

The equation for the central difference method can be solved **explicitly**.

$$\hat{k}_i u_{i+1} = p_i - f_i(u_i) - \left[\frac{m}{(\Delta t)^2} - \frac{c}{2\Delta t} \right] u_{i-1} + \frac{2m}{(\Delta t)^2} u_i$$

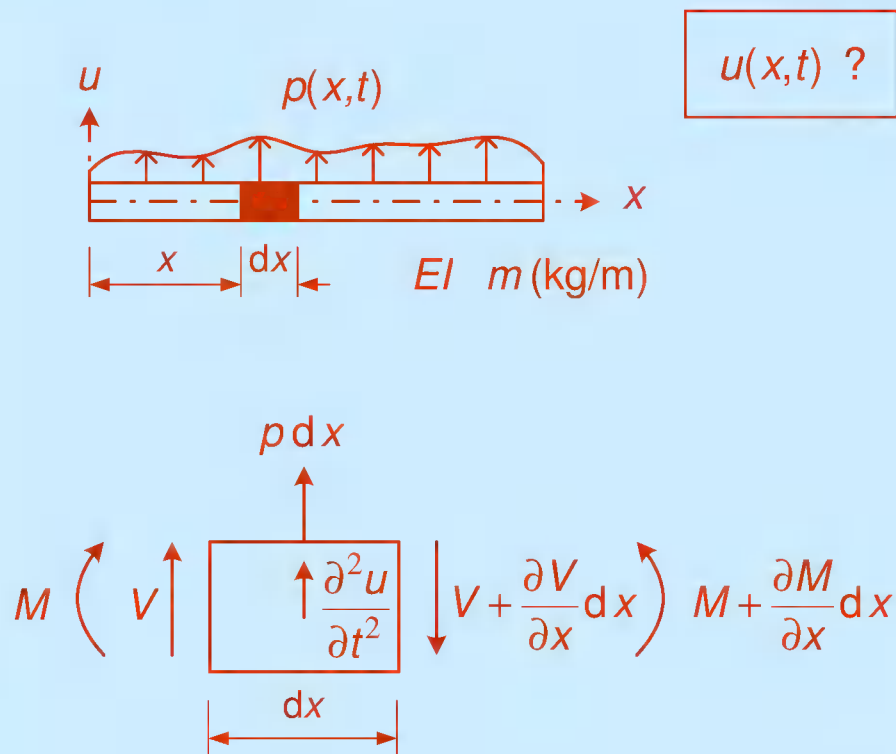
The equation for the average acceleration method must be solved by iteration (**implicitly**) since the first term in [] is unknown.

$$\left[\frac{\partial f}{\partial u} \Big|_{i+1} + \frac{2c}{\Delta t} + \frac{4m}{(\Delta t)^2} \right] u_{i+1} = \hat{p}_{i+1}$$

2D CONTINUOUS BEAMS

Free vibration in bending of a beam with distributed mass and flexibility is studied.

DIFFERENTIAL EQUATION



Vertical dynamic equation

$$m dx \frac{\partial^2 u}{\partial t^2} = V - \left(V + \frac{\partial V}{\partial x} dx \right) + p dx$$

$$\rightarrow m \frac{\partial^2 u}{\partial t^2} + \frac{\partial V}{\partial x} = p \quad (1)$$

Rotational static equation (the inertial moment associated with the angular acceleration is neglected) :

$$V = \frac{\partial M}{\partial x}$$

$$M = EI \frac{\partial^2 u}{\partial x^2} \rightarrow \frac{\partial V}{\partial x} = \frac{\partial^2 M}{\partial x^2} = EI \frac{\partial^4 u}{\partial x^4} \quad (2)$$

Equations (1) and (2) give

$$EI \frac{\partial^4 u}{\partial x^4} + m \frac{\partial^2 u}{\partial t^2} = p(t)$$

FREE VIBRATION

The differential equation

$$EI \frac{\partial^4 u}{\partial x^4} + m \frac{\partial^2 u}{\partial t^2} = 0$$

can be solved analytically.

A possible solution is of the form

$$u(x,t) = \phi(x) \cdot f(t)$$

which gives

$$EI \phi^{(4)}(x) f(t) + m \phi(x) \ddot{f}(t) = 0$$

$$\text{with } \phi^{(4)}(x) = \frac{d^4 \phi(x)}{dx^4} \quad \ddot{f}(t) = \frac{d^2 f(t)}{dt^2}$$

$$\rightarrow \frac{\phi^{(4)}(x)}{\phi(x)} = -\frac{m \ddot{f}(t)}{EI f(t)}$$

The above equation must hold for every x and t

$$\rightarrow \frac{\phi^{(4)}(x)}{\phi(x)} = -\frac{m \ddot{f}(t)}{EI f(t)} = a^4 \quad (a \text{ constant})$$

$$\rightarrow \begin{cases} \phi^{(4)}(x) - a^4 \phi(x) = 0 \\ \ddot{f}(t) + \omega^2 f(t) = 0 \end{cases}$$

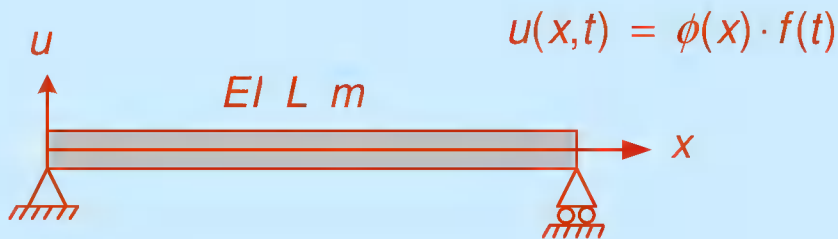
$$\omega^2 = \frac{EI a^4}{m}$$

The solution is

$$\begin{cases} f(t) = E \cos(\omega t) + F \sin(\omega t) \\ \phi(x) = A \sin(ax) + B \cos(ax) + C \sinh(ax) + D \cosh(ax) \end{cases}$$

E, F are determined by the initial conditions.

A, B, C, D are determined by the boundary conditions.

EXAMPLE: SIMPLY SUPPORTED BEAM

Boundary conditions

$$M = EI \frac{\partial^2 u}{\partial x^2} = EI \phi''(x) \cdot f(t)$$

$$\begin{cases} u(x=0,t) = 0 \\ u(x=L,t) = 0 \\ M(x=0,t) = 0 \\ M(x=L,t) = 0 \end{cases} \rightarrow \begin{cases} \phi(0) = 0 \\ \phi(L) = 0 \\ \phi''(0) = 0 \\ \phi''(L) = 0 \end{cases}$$

$$\phi(x) = A \sin(ax) + B \cos(ax) + C \sinh(ax) + D \cosh(ax)$$

$$\phi''(x) = a^2 [-A \sin(ax) - B \cos(ax) + C \sinh(ax) + D \cosh(ax)]$$

$$\begin{cases} \phi(0) = 0 \\ \phi''(0) = 0 \end{cases} \rightarrow \begin{cases} B + D = 0 \\ -B + D = 0 \end{cases} \rightarrow \begin{cases} B = 0 \\ D = 0 \end{cases}$$

$$\begin{cases} \phi(L) = 0 \\ \phi''(L) = 0 \end{cases} \rightarrow \begin{cases} A \sin(aL) + C \sinh(aL) = 0 \\ -A \sin(aL) + C \sinh(aL) = 0 \end{cases}$$

$$\rightarrow \begin{cases} A \sin(aL) = 0 & (1) \\ C \sinh(aL) = 0 & (2) \end{cases}$$

$$(2) \rightarrow C = 0 \text{ or } a = 0 \text{ (no motion)} \rightarrow C = 0$$

$$(1) \rightarrow A = 0 \text{ (no motion) or } \sin(aL) = 0$$

Conclusion: the boundary conditions require

$$\sin(aL) = 0 \text{ and } B = C = D = 0$$

$$\sin(aL) = 0 \rightarrow aL = n\pi \rightarrow a = \frac{n\pi}{L}$$

$$\omega^2 = \frac{Ela^4}{m} \rightarrow \boxed{\omega_n = n^2\pi^2 \sqrt{\frac{EI}{mL^4}}}$$

$$B = C = D = 0 \rightarrow \boxed{\phi_n(x) = A \sin\left(\frac{n\pi x}{L}\right)}$$

ω_n are the natural circular frequencies
 ϕ_n are the natural eigenmodes

An infinite number of solutions u_n have been found

$$u_n(x, t) = \phi_n(x) \cdot [E_n \cos(\omega_n t) + F_n \sin(\omega_n t)]$$

The general solution for free vibration is a superposition of all the eigenmodes.

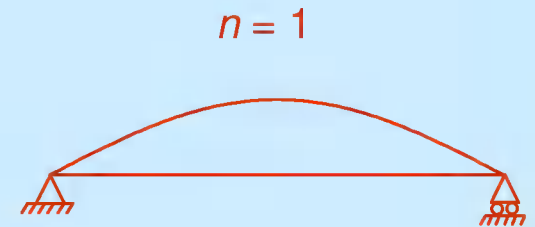
$$u(x, t) = \sum_{n=1}^{\infty} \phi_n(x) \cdot [E_n \cos(\omega_n t) + F_n \sin(\omega_n t)]$$

E_n, F_n are given by the init. cond. $u(x, t=0), \dot{u}(x, t=0)$

visualisation of the three first eigenmodes

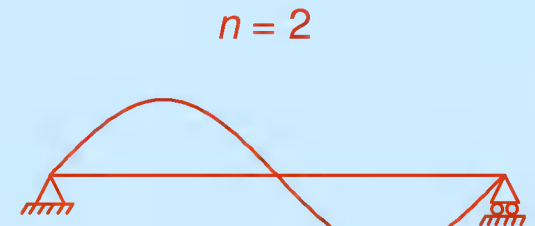
$$\omega_1 = \pi^2 \sqrt{\frac{EI}{mL^4}}$$

$$\phi_1(x) = A \sin\left(\frac{\pi x}{L}\right)$$



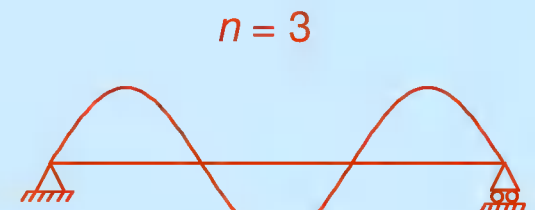
$$\omega_2 = 4\pi^2 \sqrt{\frac{EI}{mL^4}}$$

$$\phi_2(x) = A \sin\left(\frac{2\pi x}{L}\right)$$



$$\omega_3 = 9\pi^2 \sqrt{\frac{EI}{mL^4}}$$

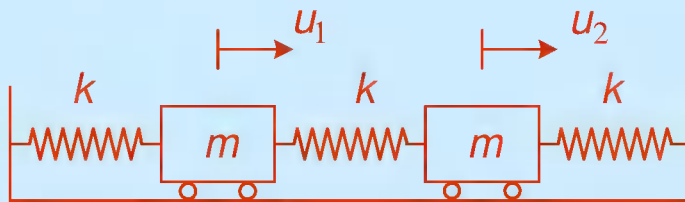
$$\phi_3(x) = A \sin\left(\frac{3\pi x}{L}\right)$$



MDF – FREE VIBRATION

Free vibration of multi degree of freedom systems are studied.

EXAMPLE 1 - TRAIN



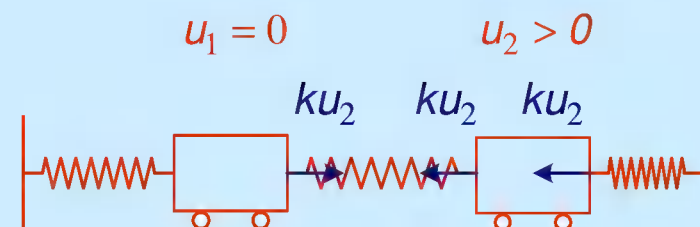
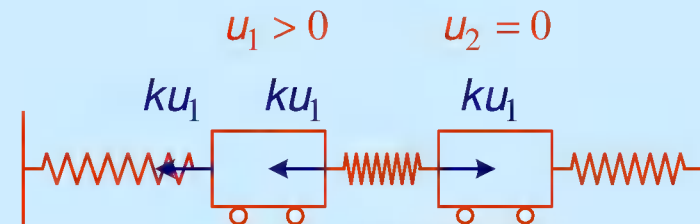
Differential equations

Newton's law is applied for each coach

$$\begin{cases} m \ddot{u}_1 = (-2k) u_1 + (k) u_2 \\ m \ddot{u}_2 = (k) u_1 + (-2k) u_2 \end{cases}$$

$$\rightarrow \begin{cases} m \ddot{u}_1 + 2k u_1 - k u_2 = 0 \\ m \ddot{u}_2 - k u_1 + 2k u_2 = 0 \end{cases}$$

The coefficients in parenthesis are determined by considering first u_1 alone and then u_2 alone.



Mass and Stiffness matrices

The differential equations can be written as

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

mass matrix

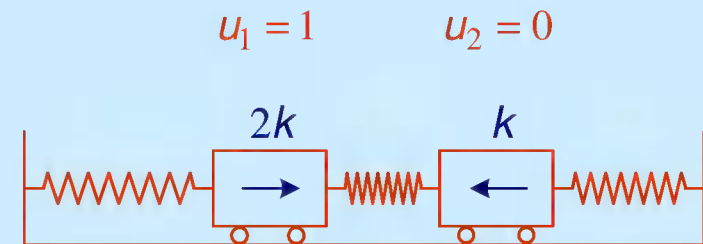
$[m]$

stiffness matrix

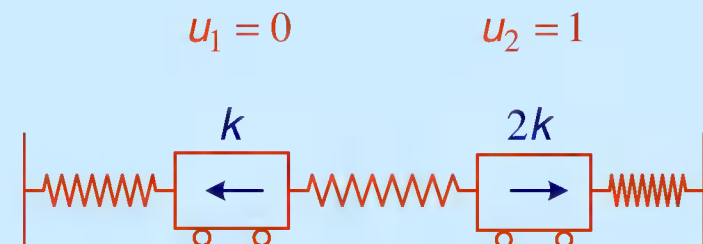
$[k]$

The stiffness matrix can be determined by the classical way. As example, the components of first column of $[k]$ are the external forces that must be applied to the masses in order to keep the system in equilibrium when a unit displacement $u_1=1$ is applied.

first column of $[k]$ $\begin{bmatrix} 2k \\ -k \end{bmatrix}$



second column of $[k]$ $\begin{bmatrix} -k \\ 2k \end{bmatrix}$



Eigenvalue problem

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The solution of this system is of the form

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} \sin(\omega_n t) \quad \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} ? \quad \omega_n ?$$

Since ϕ_1 , ϕ_2 , ω_n are constants, derivation gives

$$\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} = -\omega_n^2 \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} \sin(\omega_n t)$$

By introducing the two expressions above in the system, it is obtained

$$-\omega_n^2 [\mathbf{m}] \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} \sin(\omega_n t) + [\mathbf{k}] \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} \sin(\omega_n t) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\rightarrow ([\mathbf{k}] - \omega_n^2 [\mathbf{m}]) \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} \sin(\omega_n t) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

This equation is valid for every t , which implies

$$([\mathbf{k}] - \omega_n^2 [\mathbf{m}]) \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (1) \quad \text{eigenvalue problem}$$

This system has one trivial solution ($\phi_1 = \phi_2 = 0$) which corresponds to equilibrium (no motion).

Other solutions can be found if the following condition is respected:

$$\det([\mathbf{k}] - \omega_n^2 [\mathbf{m}]) = 0 \quad (2)$$

$$(2) \rightarrow \begin{vmatrix} 2k - \omega_n^2 m & -k \\ -k & 2k - \omega_n^2 m \end{vmatrix} = 0$$

$$\rightarrow (2k - \omega_n^2 m)^2 - k^2 = 0$$

$$\rightarrow (2k - \omega_n^2 m - k)(2k - \omega_n^2 m + k) = 0$$

$$\rightarrow \omega_n = \sqrt{\frac{k}{m}} \quad \text{or} \quad \omega_n = \sqrt{\frac{3k}{m}}$$

Two values of ω_n have been found. For each of them, ϕ_1 ϕ_2 can be determined by introducing the value of ω_n in the system (1).

$$([k] - \omega_n^2 [m]) \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (1)$$

$$\omega_1 = \sqrt{\frac{k}{m}} \quad (\omega_1^2 m = k)$$

$$(1) \rightarrow \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

This system has an infinite number of solutions, one of them is

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\omega_2 = \sqrt{\frac{3k}{m}} \quad (\omega_2^2 m = 3k)$$

$$(1) \rightarrow \begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\rightarrow \text{e.g.} \quad \begin{Bmatrix} \phi_1 \\ \phi_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Conclusion : Two solutions have been found

$$\begin{aligned} \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} &= \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \sin(\omega_1 t) \\ \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} &= \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \sin(\omega_2 t) \end{aligned}$$

$\{\phi_1\}$ ω_1
 $\{\phi_2\}$ ω_2

$\omega_1 \ \omega_2$: natural circular frequencies

$\{\phi_1\} \ \{\phi_2\}$: eigenmodes

Every linear combination of this two solutions is also a solution of the free vibration problem. The general solution can be written as

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = C_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \sin(\omega_1 t + \theta_1) + C_2 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \sin(\omega_2 t + \theta_2)$$

Physical interpretation of the eigenmodes

Free vibration is initiated by an initial deflection corresponding to eigenmode 1:

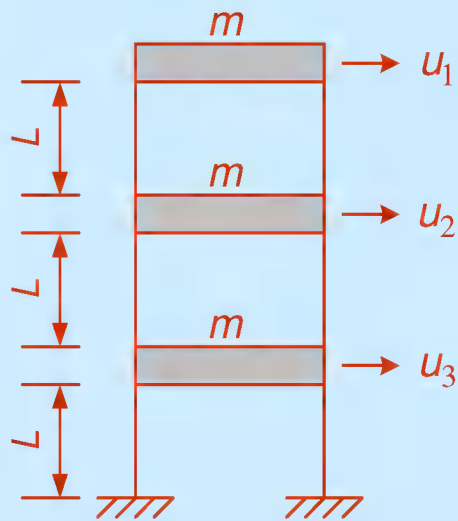
$$u_1(0) = A \quad u_2(0) = A \quad \dot{u}_1(0) = 0 \quad \dot{u}_2(0) = 0$$

With these initial conditions, the response is

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = A \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \sin(\omega_1 t)$$

- The motion of both solids is harmonic with ω_1 for circular frequency.
- The deflected shape is constant in time and corresponds to eigenmode 1.

EXAMPLE 2 – THREE STORY FRAME



$$m = 2 \text{ kg}$$

$$L = 200 \text{ mm}$$

columns

$$b = 10 \text{ mm}$$

$$h = 1 \text{ mm}$$

$$E = 200 \text{ GPa}$$

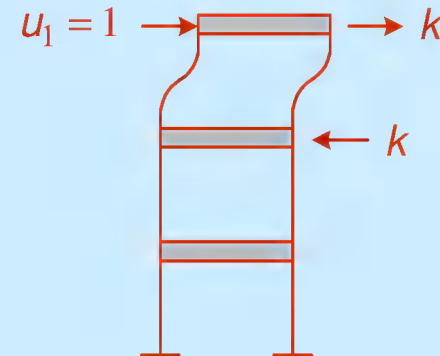
(4 columns by story)

$$k = \frac{48EI}{L^3} \quad I = \frac{bh^3}{12}$$

$$[\mathbf{m}] = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}$$

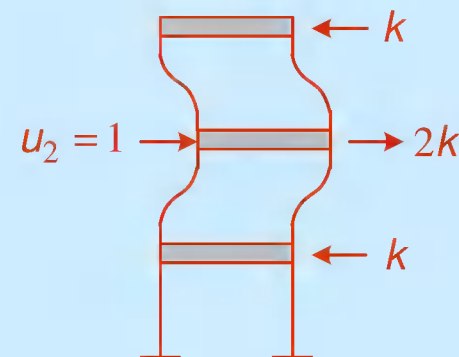
$$[\mathbf{k}] = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}$$

($[\mathbf{k}]$ is symmetric)



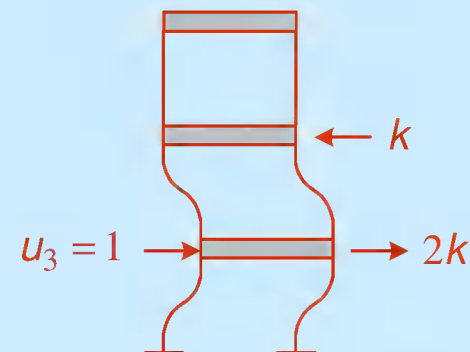
first column

$$\begin{Bmatrix} k \\ -k \\ 0 \end{Bmatrix}$$



second column

$$\begin{Bmatrix} -k \\ 2k \\ -k \end{Bmatrix}$$



third column

$$\begin{Bmatrix} 0 \\ -k \\ 2k \end{Bmatrix}$$

Eigenvalue problem

$$([k] - \omega^2 [m]) \{\phi\} = \{0\}$$

This is solved numerically by MATLAB.

$$M = [m]$$

$$K = [k]$$

$$[F, E] = \text{eig}(K, M)$$

solves the eigenvalue problem $[k]\{\phi\} = \lambda [m]\{\phi\}$

$$\rightarrow \omega^2 = \lambda$$

The MATLAB solution is given as

$$E = \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix} \quad F = \begin{bmatrix} \left\{ \phi_1 \right\} & \left\{ \phi_2 \right\} & \left\{ \phi_3 \right\} \end{bmatrix}$$

The eigenmodes are determined with an arbitrary multiplicative constant. By convention in dynamics, the eigenmodes are given on the form

$$\phi_1 = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} \quad \text{with} \quad \phi_{11} = 1$$

The eigenmodes in MATLAB are calculated such

$$\phi_1 = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} \quad \text{with} \quad \phi_{11}^2 + \phi_{21}^2 + \phi_{31}^2 = 1$$

In order to obtain the dynamic convention, the following scale command must be used.

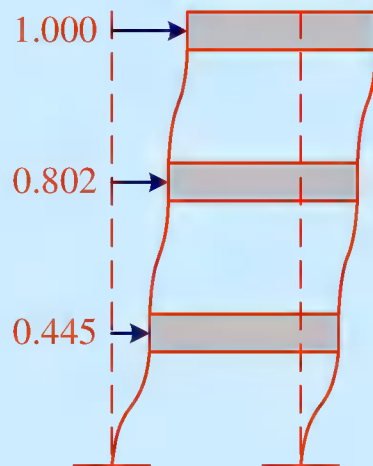
$$F = [F(:,1)/F(1,1) \quad F(:,2)/F(1,2) \quad F(:,3)/F(1,3)]$$

Results given by MATLAB

$$\omega_1 = 9.95 \text{ rad/s}$$

$$f_1 = 1.58 \text{ Hz}$$

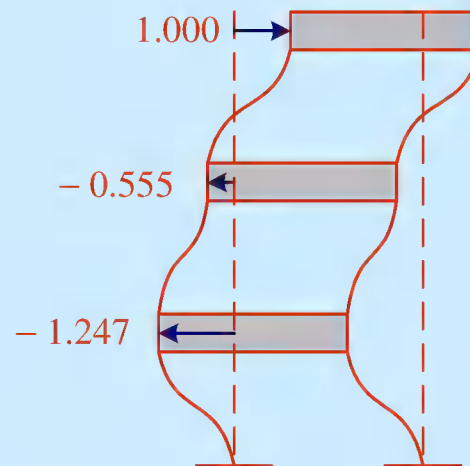
$$\phi_1 = \begin{Bmatrix} 1.000 \\ 0.802 \\ 0.445 \end{Bmatrix}$$



$$\omega_2 = 27.9 \text{ rad/s}$$

$$f_2 = 4.44 \text{ Hz}$$

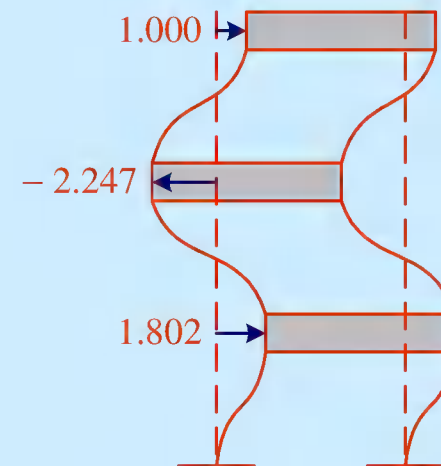
$$\phi_2 = \begin{Bmatrix} 1.000 \\ -0.555 \\ -1.247 \end{Bmatrix}$$



$$\omega_3 = 40.3 \text{ rad/s}$$

$$f_3 = 6.41 \text{ Hz}$$

$$\phi_3 = \begin{Bmatrix} 1.000 \\ -2.247 \\ 1.802 \end{Bmatrix}$$

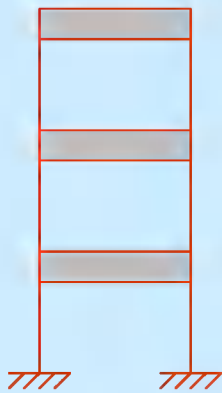


Physical interpretation of the eigenmodes

If free vibration is initiated by initial displacements corresponding to the eigenmode n , the vibration of each story will be harmonic with frequency f_n and the structure will vibrate with a constant deflected shape corresponding to the eigenmode n

Comparison between discrete and distributed systems

discrete systems



number of natural frequencies = number of degrees of freedom

eigenmode = vector

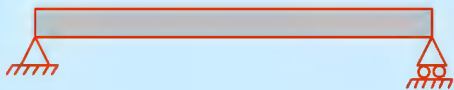
$$\phi_1 = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix}$$

the eigenmodes are defined with a multiplicative constant

convention

$$\phi_{11} = 1$$

distributed systems



infinite number of natural frequencies

eigenmode = function

$$\phi_1(x) = A \sin\left(\frac{\pi x}{L}\right)$$

convention

$$A = 1$$

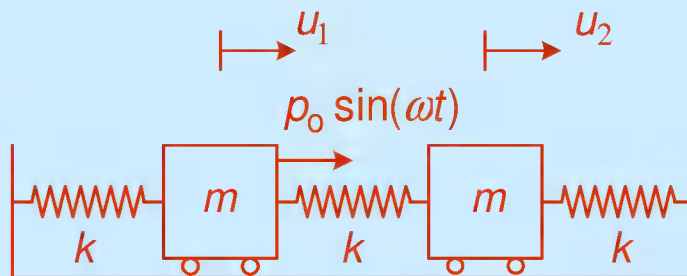
MDF – MODE ANALYSIS

The mode superposition approach is used in order to calculate the response of a MDF system to an applied load.

First harmonic loads and then arbitrary loads are considered. Finally damping is introduced.

HARMONIC LOAD – TRAIN

The first coach is excited by a harmonic load



Summary of the eigenvalue problem

$$[\mathbf{m}] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad [\mathbf{k}] = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}$$

$$\text{eigenvalue problem} \quad ([\mathbf{k}] - \omega_n^2 [\mathbf{m}]) \{\phi\} = \{0\}$$

$$\omega_1 = \sqrt{k/m} \quad \{\phi_1\} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \quad \omega_2 = \sqrt{3k/m} \quad \{\phi_2\} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

The matrix $[\phi]$ is defined as

$$[\phi] = [\{\phi_1\} \{\phi_2\}] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Differential equations

$$[\mathbf{m}] \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + [\mathbf{k}] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_0 \\ 0 \end{Bmatrix} \sin(\omega t)$$

$$\begin{cases} m \ddot{u}_1 + 2k u_1 - k u_2 = p_0 \sin(\omega t) \\ m \ddot{u}_2 - k u_1 + 2k u_2 = 0 \end{cases}$$

The two equations are coupled and it is hence not possible to solve directly the system.

Modal coordinates

The idea is to change coordinates (unknowns) and calculate first $q_1(t)$ and $q_2(t)$ which are defined by

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = [\phi] \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}$$

by derivation, it is obtained

$$\begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \end{Bmatrix} = [\phi] \begin{Bmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{Bmatrix}$$

Introducing these two expressions in the differential equations and multiplying on the left side by $[\phi]^T$ gives

$$[\phi]^T [\mathbf{m}] [\phi] \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + [\phi]^T [\mathbf{k}] [\phi] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = [\phi]^T \{\mathbf{p}\} \sin(\omega t)$$

which can be rewritten as

$$[\mathbf{M}] \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + [\mathbf{K}] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \{\mathbf{P}\} \sin(\omega t)$$

$$[\mathbf{M}] = [\phi]^T [\mathbf{m}] [\phi]$$

$$[\mathbf{K}] = [\phi]^T [\mathbf{k}] [\phi]$$

$$\{\mathbf{P}\} = [\phi]^T \{\mathbf{p}\}$$

After calculations it is obtained

$$[\mathbf{M}] = \begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix} \quad [\mathbf{K}] = \begin{bmatrix} 2k & 0 \\ 0 & 6k \end{bmatrix} \quad \{\mathbf{P}\} = \begin{Bmatrix} p_o \\ p_o \end{Bmatrix}$$

$$\begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & 6k \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} p_o \\ p_o \end{Bmatrix} \sin(\omega t)$$

$$\begin{cases} 2m \ddot{q}_1 + 2k q_1 = p_o \sin(\omega t) \\ 2m \ddot{q}_2 + 6k q_2 = p_o \sin(\omega t) \end{cases}$$

Since Both $[\mathbf{M}]$ and $[\mathbf{K}]$ are diagonal matrices, the two equations are uncoupled and can be solved separately.

Each equation represents a SDF system.

The natural circular frequencies for the first and second equations are ω_1 and ω_2 which have been calculated by the eigenvalue problem.

Each equation are solved by considering only the steady state response

$$\begin{cases} q_1(t) = \frac{p_o/2k}{1 - (\omega/\omega_1)^2} \sin(\omega t) & \omega_1 = \sqrt{k/m} \\ q_2(t) = \frac{p_o/6k}{1 - (\omega/\omega_2)^2} \sin(\omega t) & \omega_2 = \sqrt{3k/m} \end{cases}$$

u_1 and u_2 can now be calculated with

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = [\Phi] \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}$$

$$\begin{cases} u_1(t) = \left(\frac{p_o/2k}{1 - (\omega/\omega_1)^2} + \frac{p_o/6k}{1 - (\omega/\omega_2)^2} \right) \sin(\omega t) \\ u_2(t) = \left(\frac{p_o/2k}{1 - (\omega/\omega_1)^2} - \frac{p_o/6k}{1 - (\omega/\omega_2)^2} \right) \sin(\omega t) \end{cases}$$

Conclusion

$$\begin{cases} u_1(t) = \left(\frac{p_0/2k}{1 - (\omega/\omega_1)^2} + \frac{p_0/6k}{1 - (\omega/\omega_2)^2} \right) \sin(\omega t) \\ u_2(t) = \left(\frac{p_0/2k}{1 - (\omega/\omega_1)^2} - \frac{p_0/6k}{1 - (\omega/\omega_2)^2} \right) \sin(\omega t) \end{cases}$$

After some time (steady state response), the structure vibrates with the same frequency as the force.

Resonance

the load frequency ω is close to the first natural frequency of the structure ω_1

$$\omega \rightarrow \omega_1 \quad \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \frac{p_0/2k}{1 - (\omega/\omega_1)^2} \sin(\omega t)$$

- the structure vibrates with a deflected shape corresponding to eigenmode 1.
- the amplitudes of the vibrations become infinite (undamped case)

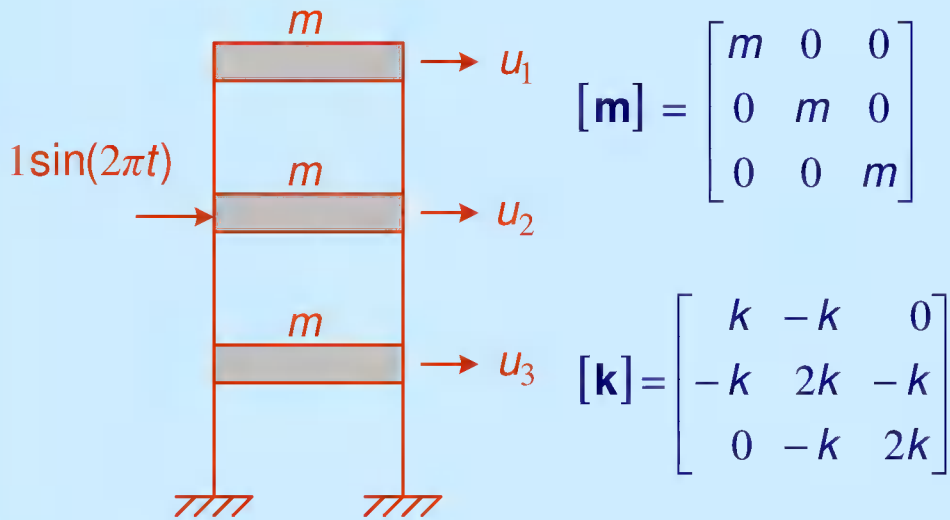
If a MDF structure is excited by a harmonic force whose the frequency is one of the natural frequencies of the structure , then after some while (steady state response)

- the structure vibrates with the same frequency as the applied force
- the deflected shape is the eigenmode associated to the natural frequency
- the amplitudes of the vibrations are large

This phenomenon is called **resonance**

HARMONIC LOAD – 3 STORY FRAME

A harmonic force is applied to the second story.



Eigenvalue analysis

$$\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} 9.951 \\ 27.88 \\ 40.29 \end{Bmatrix} \quad [\boldsymbol{\phi}] = \begin{bmatrix} 1.000 & 1.000 & 1.000 \\ 0.802 & -0.555 & -2.247 \\ 0.445 & -1.247 & 1.802 \end{bmatrix}$$

Applied harmonic force

$$\begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \sin(2\pi t) = \{\mathbf{p}\} \sin(2\pi t)$$

Mode superposition

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = [\boldsymbol{\phi}] \begin{Bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{Bmatrix}$$

$$[\mathbf{m}] \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + [\mathbf{k}] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \{\mathbf{p}\} \sin(2\pi t)$$

$$\rightarrow [\mathbf{M}] \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + [\mathbf{K}] \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \{\mathbf{P}\} \sin(2\pi t)$$

$$[\mathbf{M}] = [\phi]^T [\mathbf{m}] [\phi] = \begin{bmatrix} 3.682 & 0 & 0 \\ 0 & 5.726 & 0 \\ 0 & 0 & 18.59 \end{bmatrix}$$

$$[\mathbf{K}] = [\phi]^T [\mathbf{k}] [\phi] = \begin{bmatrix} 364.7 & 0 & 0 \\ 0 & 4452 & 0 \\ 0 & 0 & 30184 \end{bmatrix}$$

$$\{\mathbf{P}\} = [\phi]^T \{\mathbf{p}\} = \begin{Bmatrix} 0.802 \\ -0.555 \\ -2.247 \end{Bmatrix}$$

The uncoupled system is

$$\begin{cases} 3.682 \ddot{q}_1 + 364.7 q_1 = 0.802 \sin(2\pi t) \\ 5.726 \ddot{q}_2 + 4452 q_2 = -0.555 \sin(2\pi t) \\ 18.59 \ddot{q}_3 + 30184 q_3 = -2.247 \sin(2\pi t) \end{cases}$$

Each SDF system is solved by only considering the steady state response, which directly gives

$$q_1(t) = \frac{0.802/364.7}{1 - (2\pi/9.95)^2} \sin(2\pi t) = 3.656 \cdot 10^{-3} \sin(2\pi t)$$

$$q_2(t) = \frac{-0.555/4452}{1 - (2\pi/27.9)^2} \sin(2\pi t) = -1.313 \cdot 10^{-4} \sin(2\pi t)$$

$$q_3(t) = \frac{-2.247/30184}{1 - (2\pi/40.3)^2} \sin(2\pi t) = -7.629 \cdot 10^{-5} \sin(2\pi t)$$

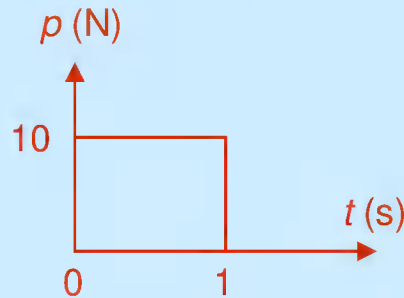
Transformation to real coordinates

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = [\phi] \begin{Bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{Bmatrix} = [\phi] \begin{Bmatrix} 3.656 \cdot 10^{-3} \\ -1.313 \cdot 10^{-4} \\ -7.629 \cdot 10^{-5} \end{Bmatrix} \sin(2\pi t)$$

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = \begin{Bmatrix} 3.449 \cdot 10^{-3} \\ 3.177 \cdot 10^{-3} \\ 1.653 \cdot 10^{-3} \end{Bmatrix} \sin(2\pi t)$$

ARBITRARY LOADING – TRAIN

$p(t)$ is applied to the first solid



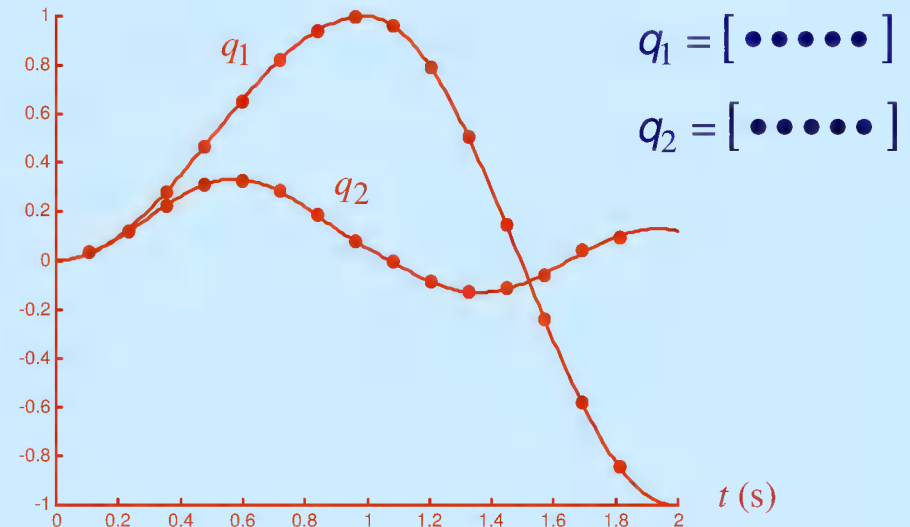
$$[\mathbf{m}] \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + [\mathbf{k}] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p(t) \\ 0 \end{Bmatrix}$$

Transformation to modal coordinates

$$[\mathbf{M}] \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + [\mathbf{K}] \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \{\mathbf{P}\}$$

$$\rightarrow \begin{cases} 2m \ddot{q}_1 + 2k q_1 = p(t) \\ 2m \ddot{q}_2 + 6k q_2 = p(t) \end{cases}$$

Each SDF equation is solved by using a time integration method (lesson 4). For this particular case, the exact solution can be found (lesson 3).



Transformation to real coordinates

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = [\Phi] \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} \quad \text{must be performed for each time step}$$

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} ***** \\ ***** \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{Bmatrix} \bullet \bullet \bullet \bullet \bullet \\ \bullet \bullet \bullet \bullet \bullet \end{Bmatrix}$$

SYSTEMS WITH DAMPING

The equation system of MDF systems with viscous damping is

$$[\mathbf{m}]\{\ddot{\mathbf{u}}\} + [\mathbf{c}]\{\dot{\mathbf{u}}\} + [\mathbf{k}]\{\mathbf{u}\} = \{\mathbf{p}(t)\}$$

The transformation to modal coordinates can be performed as before, which gives

$$\{\mathbf{u}\} = [\boldsymbol{\phi}]\{\mathbf{q}\} \quad [\mathbf{M}]\{\ddot{\mathbf{q}}\} + [\mathbf{C}]\{\dot{\mathbf{q}}\} + [\mathbf{K}]\{\mathbf{q}\} = \{\mathbf{P}(t)\}$$

$$\text{with } [\mathbf{C}] = [\boldsymbol{\phi}]^T [\mathbf{c}] [\boldsymbol{\phi}]$$

Problems :

- how to define $[\boldsymbol{\phi}]$?
- how to obtain by experiments $[\mathbf{C}]$ or $[\mathbf{c}]$?
- useful transformation only if $[\mathbf{C}]$ is diagonal

Solution :

1) Natural frequencies and modes are calculated by neglected damping. This is correct if the damping is small.

2) Transformation to normal coordinates : a damping coefficient ξ_n is introduced for each mode. ξ_n is determined by experiment. This gives n uncoupled equations on the form.

$$M_n \ddot{q}_n + C_n \dot{q}_n + K_n q_n = P_n(t) \quad \xi_n = \frac{C_n}{2\sqrt{K_n M_n}}$$

3) Each SDF equation is solved by time integration method

$$\rightarrow \{\mathbf{q}(t)\}$$

4) Transformation to real coordinates

$$\{\mathbf{u}\} = [\boldsymbol{\phi}]\{\mathbf{q}\}$$

DAMPING MATRIX

For certain problems, it is better to solve directly the coupled differential equations

$$[\mathbf{m}]\{\ddot{\mathbf{u}}\} + [\mathbf{c}]\{\dot{\mathbf{u}}\} + [\mathbf{k}]\{\mathbf{u}\} = \{\mathbf{p}(t)\}$$

instead of using the modal transformation (see next section). The damping matrix $[\mathbf{c}]$ is then needed.

$[\mathbf{c}]$ is calculated from the modal damping ratios ξ_n which are determined by experiments.

Method one

$[\mathbf{c}]$ is calculated from the modal matrix $[\mathbf{C}]$

$$\xi_n = \frac{C_n}{2\sqrt{K_n M_n}} \rightarrow [\mathbf{C}] = \begin{bmatrix} C_1 & & \\ & C_2 & \\ & & C_3 \end{bmatrix}$$

$$[\mathbf{C}] = [\boldsymbol{\phi}]^T [\mathbf{c}] [\boldsymbol{\phi}] \rightarrow [\mathbf{c}] = ([\boldsymbol{\phi}]^T)^{-1} [\mathbf{C}] [\boldsymbol{\phi}]$$

This method supposes that all the damping ratios ξ_n are known, which is usually not the case.

Method two : Rayleigh damping

The damping matrix is taken as

$$[\mathbf{c}] = a_0 [\mathbf{m}] + a_1 [\mathbf{k}]$$

It can easily be shown that the modal coordinates transformation leads to a diagonal modal damping matrix and that

$$\xi_n = \frac{a_0}{2} \frac{1}{\omega_n} + \frac{a_1}{2} \omega_n$$

Hence, $[\mathbf{c}]$ is calculated using only two modal damping coefficients.

The two modes with specified ξ_n should be chosen to ensure reasonable values for the other damping ratios. In practice, the lowest modes and the third or fourth lowest ones are used to determine a_0 and a_1 .

EFFECT OF THE DAMPING ON HARMONIC VIBRATIONS – TRAIN

The first solid is excited by a force $p_0 \sin(\omega t)$

Transformation to modal coordinates

$$\begin{cases} 2m \ddot{q}_1 + C_1 \dot{q}_1 + 2k q_1 = p_0 \sin(\omega t) \\ 2m \ddot{q}_2 + C_2 \dot{q}_2 + 6k q_2 = p_0 \sin(\omega t) \end{cases}$$

The steady state response for each equation is on the form

$$\begin{cases} q_1(t) = D_1 \sin(\omega t - \theta_1) \\ q_2(t) = D_2 \sin(\omega t - \theta_2) \end{cases}$$

$$D_1 = \frac{p_0 / 2k}{\sqrt{[1 - (\omega/\omega_1)^2]^2 + [2\xi_1(\omega/\omega_1)]^2}}$$

$$\tan(\theta_1) = \frac{2\xi_1(\omega/\omega_1)}{1 - (\omega/\omega_1)^2}$$

Transformation to real coordinates : after some work, the response can be written as

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = [\Phi] \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{Bmatrix} G_1 \sin(\omega t - \alpha_1) \\ G_2 \sin(\omega t - \alpha_2) \end{Bmatrix}$$

Conclusion :

- Both solids vibrate with the same frequency as the applied force, but with a certain difference of phase. It means that $u_1(t)$ and $u_2(t)$ don't reach their maximal values at the same time.
- The resonance properties enounced without damping are still valid.

MDF - TIME INTEGRATION METHODS

The system of differential equations can also be solved by time stepping methods. These methods are similar to the ones for SDF systems (lesson 4) by replacing scalars by matrices and vectors.

$$[\mathbf{m}]\{\ddot{\mathbf{u}}\} + [\mathbf{c}]\{\dot{\mathbf{u}}\} + [\mathbf{k}]\{\mathbf{u}\} = \{\mathbf{p}(t)\}$$

Initial conditions $\{\mathbf{u}\}_0 \quad \{\dot{\mathbf{u}}\}_0$

CENTRAL DIFFERENCE METHOD

(explicit algorithm)

This method is based on two difference expressions and the equilibrium at time i .

$$\{\dot{\mathbf{u}}\}_i = \frac{\{\mathbf{u}\}_{i+1} - \{\mathbf{u}\}_{i-1}}{2\Delta t} \quad \{\ddot{\mathbf{u}}\}_i = \frac{\{\mathbf{u}\}_{i+1} - 2\{\mathbf{u}\}_i + \{\mathbf{u}\}_{i-1}}{(\Delta t)^2}$$

$$[\mathbf{m}]\{\ddot{\mathbf{u}}\}_i + [\mathbf{c}]\{\dot{\mathbf{u}}\}_i + [\mathbf{k}]\{\mathbf{u}\}_i = \{\mathbf{p}\}_i$$

combination of these 3 eqs. gives

$$\left[\frac{[\mathbf{m}]}{(\Delta t)^2} + \frac{[\mathbf{c}]}{2\Delta t} \right] \{\mathbf{u}\}_{i+1} = \{\mathbf{p}\}_i - \left[\frac{[\mathbf{m}]}{(\Delta t)^2} - \frac{[\mathbf{c}]}{2\Delta t} \right] \{\mathbf{u}\}_{i-1} - \left[[\mathbf{k}] - \frac{2[\mathbf{m}]}{(\Delta t)^2} \right] \{\mathbf{u}\}_i$$

using the following approximation

$$\frac{1}{2\Delta t}[\mathbf{c}](\{\mathbf{u}\}_{i-1} - \{\mathbf{u}\}_{i+1}) \approx \frac{1}{\Delta t}[\mathbf{c}](\{\mathbf{u}\}_{i-1} - \{\mathbf{u}\}_i)$$

it is obtained

$$\frac{[\mathbf{m}]}{(\Delta t)^2} \{\mathbf{u}\}_{i+1} = \{\mathbf{p}\}_i - \left[\frac{[\mathbf{m}]}{(\Delta t)^2} - \frac{[\mathbf{c}]}{2\Delta t} \right] \{\mathbf{u}\}_{i-1} - \left[[\mathbf{k}] - \frac{2[\mathbf{m}]}{(\Delta t)^2} + \frac{[\mathbf{c}]}{\Delta t} \right] \{\mathbf{u}\}_i$$

Then, if $[\mathbf{m}]$ is diagonal, the solution of simultaneous equations is not required.

AVERAGE ACCELERATION METHOD

(implicit algorithm)

For this method, a set of uncoupled equations cannot be obtained by simplification since the equations are on the form

$$\left[[\mathbf{k}] + \frac{2[\mathbf{c}]}{\Delta t} + \frac{4[\mathbf{m}]}{(\Delta t)^2} \right] \{\mathbf{u}\}_{i+1} = \{\hat{\mathbf{p}}\}_i$$

And if $[m]$ can often be taken as diagonal, this is not possible for the stiffness matrix $[k]$

CONCLUDING REMARKS

The modal superposition requires that the system is linear, i.e. small elastic deformations.

By comparison to time integration methods, the main interest of the modal analysis is that in most of structural dynamics problems, only a few modal contributions need to be solved.

Example

$$[\mathbf{m}]\{\ddot{\mathbf{u}}\} + [\mathbf{c}]\{\dot{\mathbf{u}}\} + [\mathbf{k}]\{\mathbf{u}\} = \{\mathbf{p}(t)\} \quad 10000 \text{ d.o.f.}$$

→ 10000 modal equations

$$M_n \ddot{q}_n + C_n \dot{q}_n + K_n q_n = P_n(t)$$

Only the 10 first equations, corresponding to the 10 lowest natural frequencies need to be solved. $q_{11} \dots q_{10000}$ can be neglected and don't need to be calculated.

$$\begin{Bmatrix} u_1(t) \\ \dots \\ u_{10}(t) \\ u_{11}(t) \\ \dots \\ u_{10000}(t) \end{Bmatrix} \approx [\phi] \begin{Bmatrix} q_1(t) \\ \dots \\ q_{10}(t) \\ 0 \\ \dots \\ 0 \end{Bmatrix}$$

$$\{\mathbf{u}(t)\} \approx \{\phi_1\}q_1(t) + \{\phi_2\}q_2(t) \dots + \{\phi_{10}\}q_{10}(t)$$

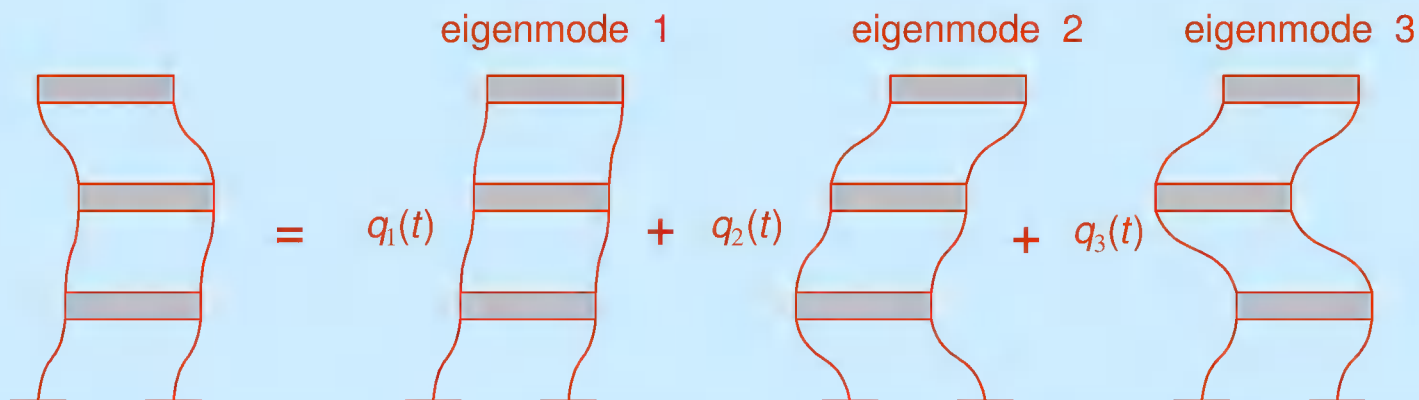
Implicit time integration methods are suited when many modes would be needed in the modal method. These methods are often unconditionally stable and the time step Δt is only limited by accuracy.

Explicit time integration methods are often used in combination with diagonal mass matrices which allows a small cost per time step. These methods are often conditionally stable and the time step must be very small so that stability is ensured for all the modes: even if the response in the higher modes is insignificant, it will blow up if the stability requirements are not satisfied relative to these modes.

Contrary to implicit methods, explicit methods do not require an iteration process for nonlinear problems.

INTERPRETATION OF THE MODAL SUPERPOSITION

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = [\Phi] \begin{Bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{Bmatrix} = \begin{Bmatrix} \phi_{11} \\ \phi_{12} \\ \phi_{13} \end{Bmatrix} q_1(t) + \begin{Bmatrix} \phi_{21} \\ \phi_{22} \\ \phi_{23} \end{Bmatrix} q_2(t) + \begin{Bmatrix} \phi_{31} \\ \phi_{32} \\ \phi_{33} \end{Bmatrix} q_3(t)$$

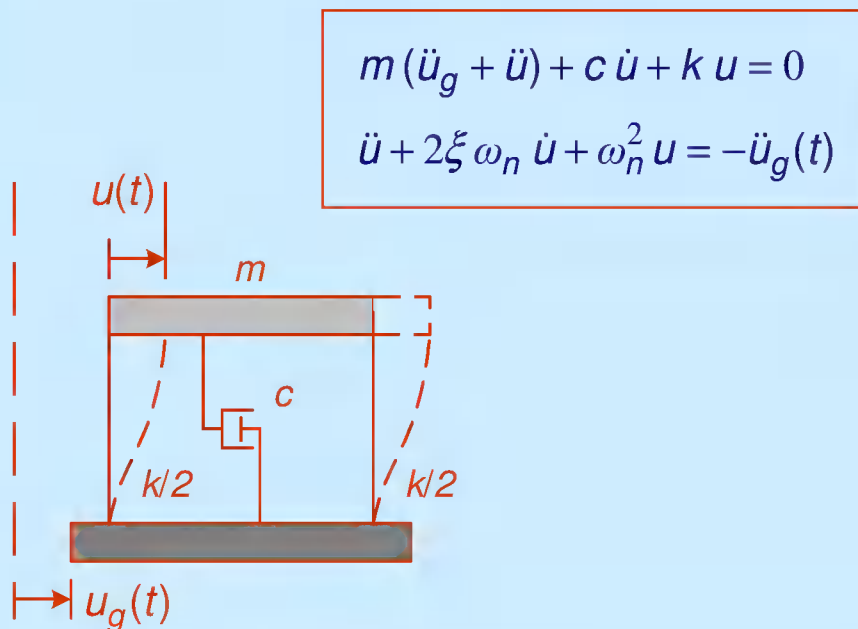


EARTHQUAKE ANALYSIS

An introduction to earthquake analysis is given.
Analysis by response spectra is presented.

SDF SYSTEMS

A SDF system is subjected to a ground motion $u_g(t)$. The deformation response $u(t)$ is to be calculated.



The ground acceleration can be registered using accelerographs, see the example below.

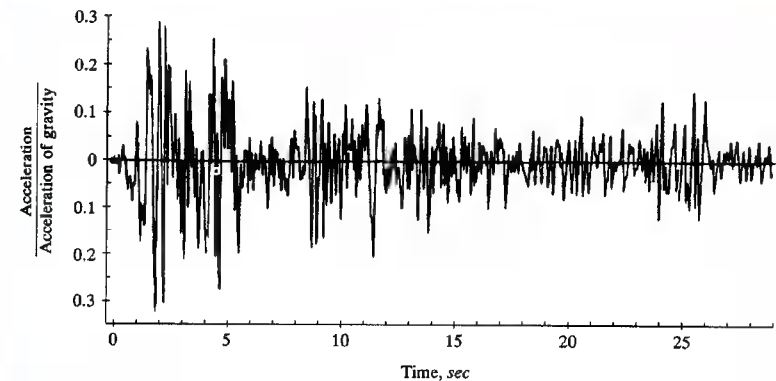
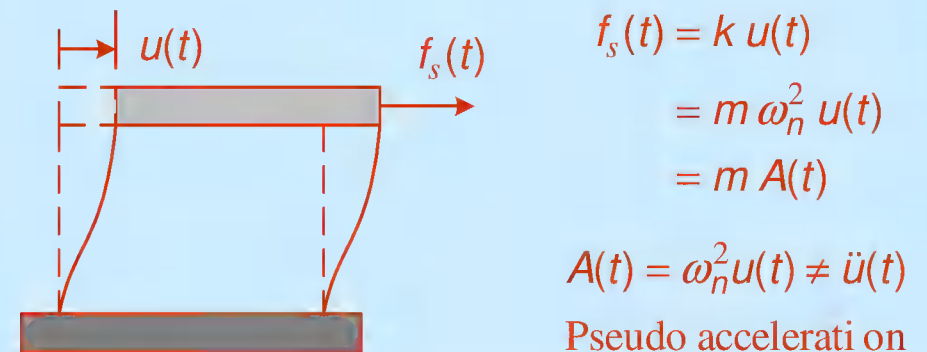


FIGURE 24-15
Accelerogram from El Centro earthquake, May 18, 1940 (NS component).

EQUIVALENT STATIC FORCE



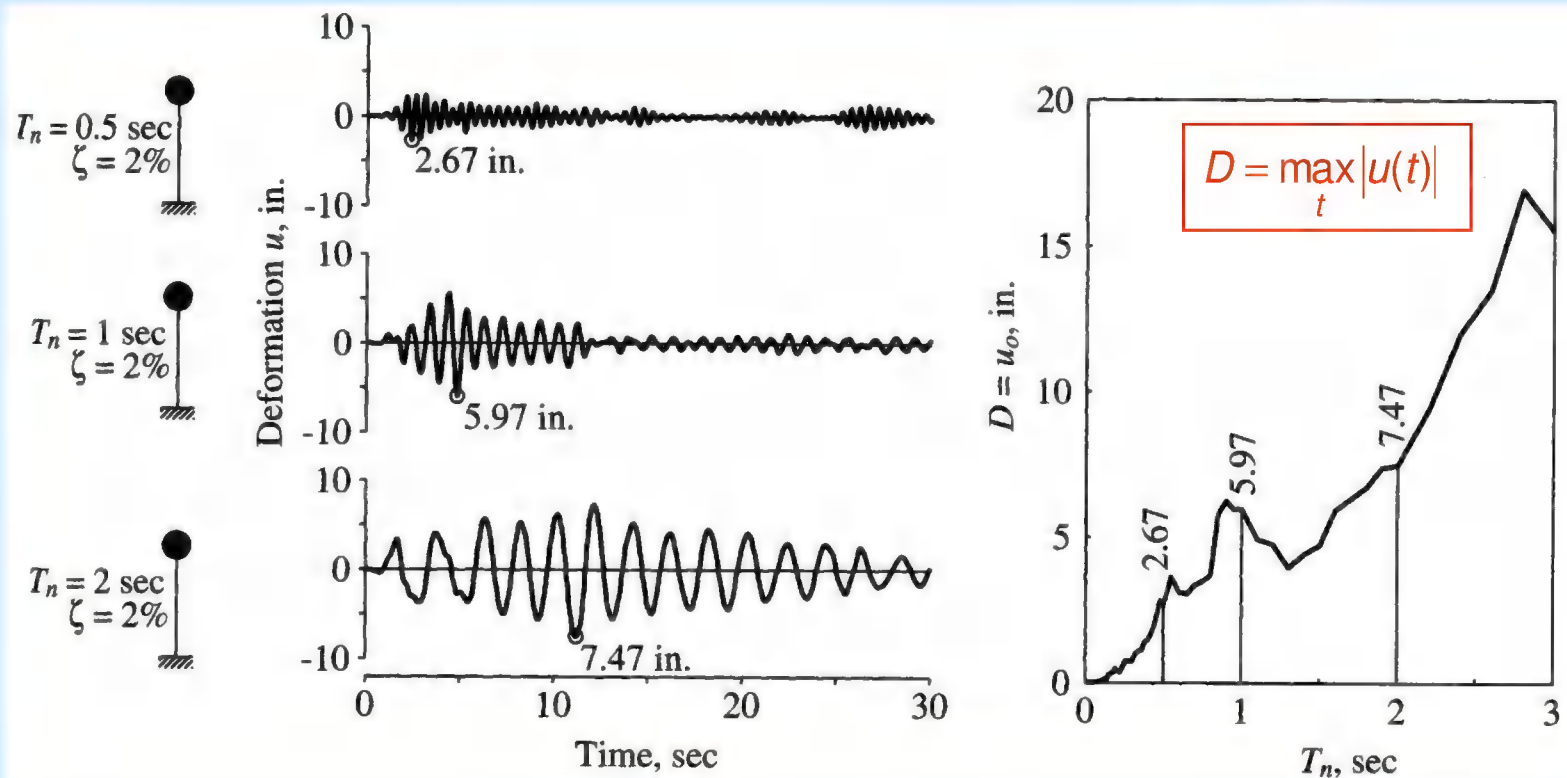
$f_s(t)$ is the force which must be applied statically in order to create a displacement $u(t)$.

RESPONSE SPECTRA

A response spectrum is a plot of maximum response (e.g. displacement, velocity, acceleration) of SDF systems to a given ground acceleration versus systems parameters (T_n , ξ).

A response spectrum is calculated numerically using time integration methods for many values of parameters (T_n , ξ).

Example : Deformation response spectrum for El Centro earthquake



Deformation, pseudo-velocity and **pseudo-acceleration** response spectra can be defined and plotted on the same graphs

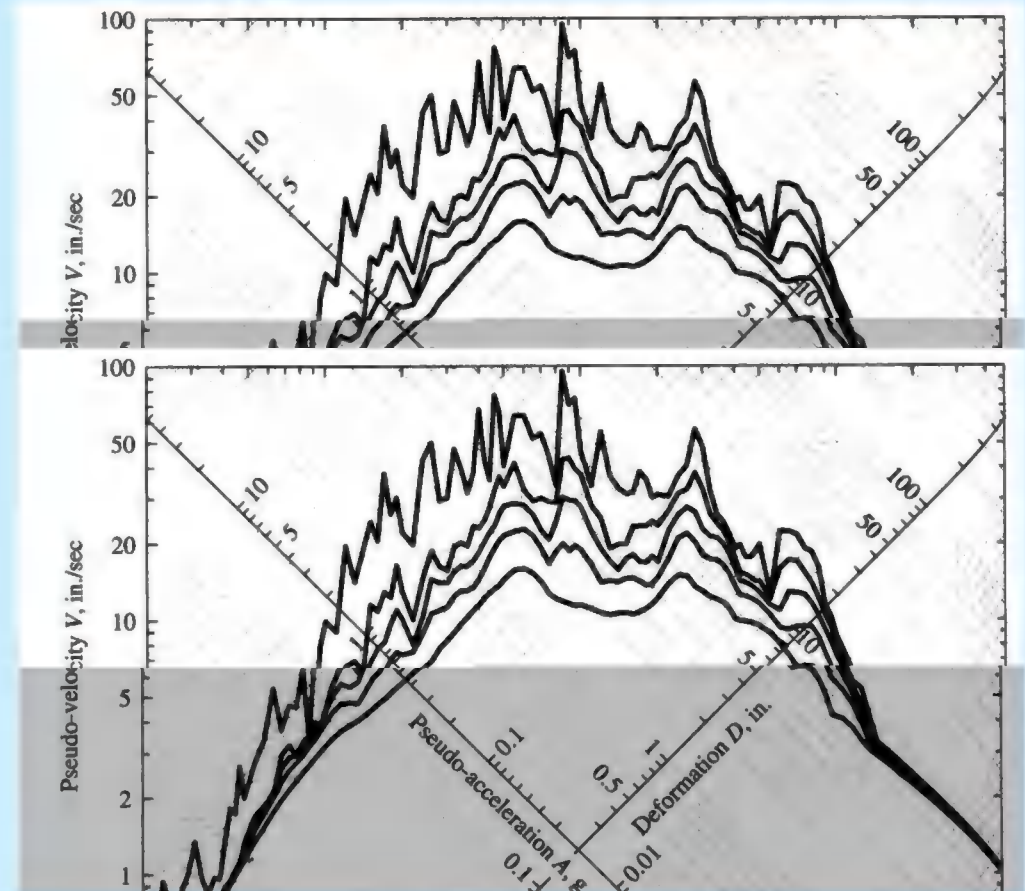
Peak Deformation $D = \max |u(t)|$

Peak Pseudo-velocity $V = \omega_n D$

Peak Pseudo-acceleration $A = \omega_n^2 D$

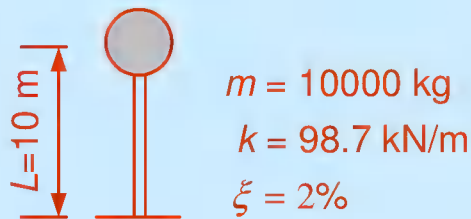
ω_n : natural circular frequency
of the SDF system.

COMBINED D-V-A SPECTRUM



EXAMPLE

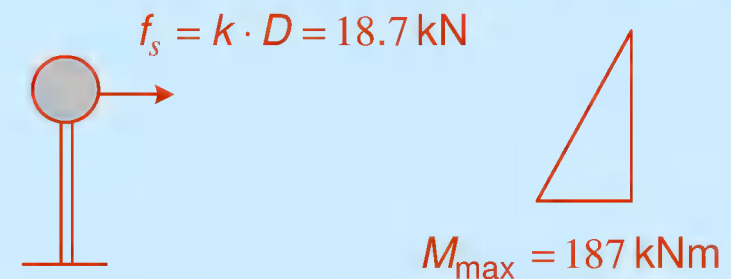
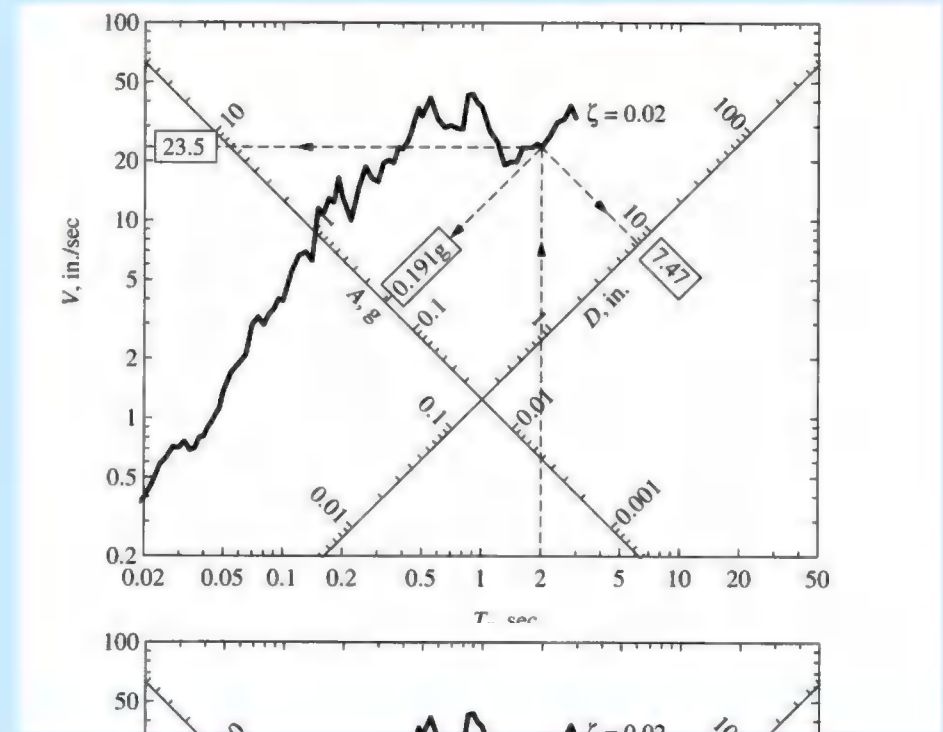
A water tank is subjected to the El Centro earthquake. Calculate the maximum bending moment during the earthquake.



$$\omega_n = \sqrt{\frac{k}{m}} = 3.14 \text{ rad/s} \rightarrow T_n = \frac{2\pi}{\omega_n} = 2 \text{ s}$$

$$\text{Spectrum} \rightarrow \begin{cases} D = 7.47 \cdot 25.4 = 190 \text{ mm} \\ A = 0.191 \cdot 9.81 = 1.87 \text{ ms}^{-2} \end{cases}$$

$$(\text{obs} : A = \omega_n^2 D)$$



When the equivalent static force has been determined, the internal forces and stresses can be determined using statics.

RESPONSE SPECTRUM CHARACTERISTICS

General characteristics can be derived from the analysis of response spectra.

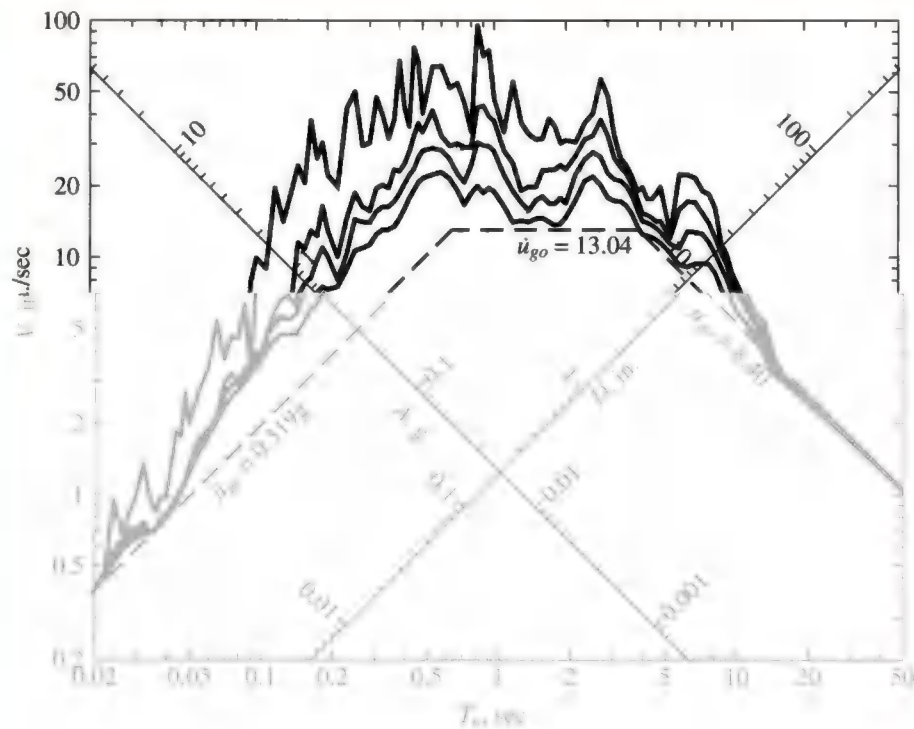


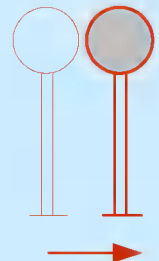
Figure 6.8.1 Response spectrum ($\zeta = 0, 2, 5, \text{ and } 10\%$) and peak values of ground acceleration, ground velocity, and ground displacement for El Centro ground motion.

$$T_n = 2\pi\sqrt{m/k}$$

$T_n < 0.03 \text{ s}$: rigid system

no deformation

$$u(t) \approx 0 \rightarrow D \approx 0$$



$T_n > 15 \text{ s}$: flexible system

no total displacement

$$u(t) = u_g(t) \rightarrow D = u_{go}$$



The spectrum can be divided in 3 period ranges :

$T_n < 0.5 \text{ s}$: acceleration sensitive region

$0.5 < T_n < 3 \text{ s}$: velocity sensitive region

$T_n > 3 \text{ s}$: displacement sensitive region

ELASTIC DESIGN SPECTRUM

Problem: how to ensure that a structure will resist future earthquakes.

The elastic design spectrum is obtained from ground motions data recorded during past earthquakes at the site or in regions with near-similar conditions

EXAMPLE

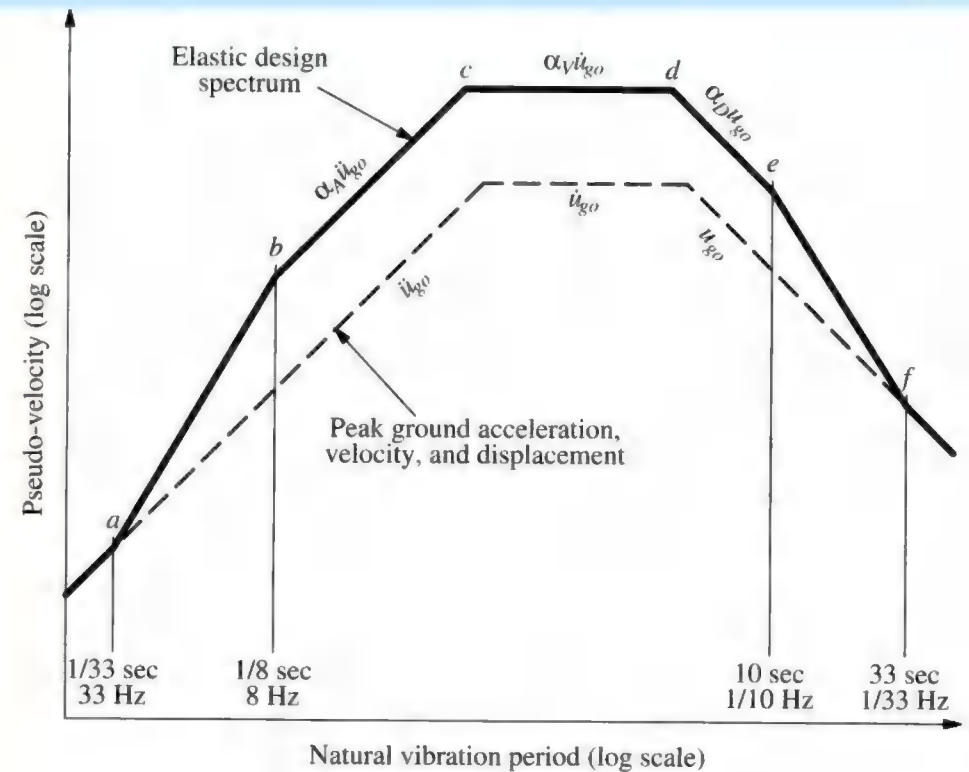
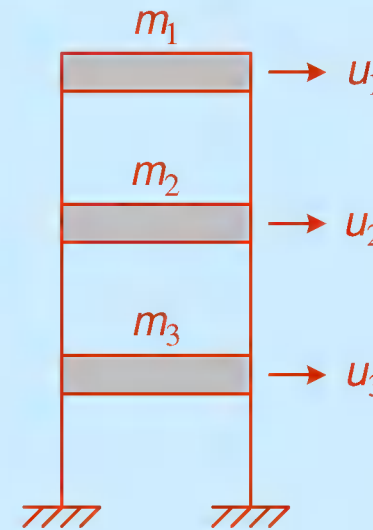


Figure 6.9.3 Construction of elastic design spectrum.

MDF SHEAR BUILDING

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 + \ddot{u}_g \\ \ddot{u}_2 + \ddot{u}_g \\ \ddot{u}_3 + \ddot{u}_g \end{Bmatrix} + [\mathbf{c}] \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} + [\mathbf{k}] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + [\mathbf{c}] \begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{Bmatrix} + [\mathbf{k}] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = - \begin{Bmatrix} m_1 \\ m_2 \\ m_3 \end{Bmatrix} \ddot{u}_g(t)$$



Eigenvalue analysis

$$\omega_1 \quad \omega_2 \quad \omega_3 \quad \phi_1 = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{Bmatrix} \quad [\phi] = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix}$$

Modal transformation

$$\begin{bmatrix} M_1 & & \\ & M_2 & \\ & & M_3 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} C_1 & & \\ & C_2 & \\ & & C_3 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} + \begin{bmatrix} K_1 & & \\ & K_2 & \\ & & K_3 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = -[\phi]^T \begin{Bmatrix} m_1 \\ m_2 \\ m_3 \end{Bmatrix} \ddot{u}_g(t) \quad \{P\} = [\phi]^T \begin{Bmatrix} m_1 \\ m_2 \\ m_3 \end{Bmatrix}$$

$$M_n \ddot{q}_n + C_n \dot{q}_n + K_n q_n = -P_n \ddot{u}_g(t) \quad n = 1, 2, 3$$

$$\ddot{q}_n + 2\xi_n \omega_n \dot{q}_n + \omega_n^2 q_n = -\Gamma_n \ddot{u}_g(t)$$

$$\Gamma_n = \frac{P_n}{M_n}$$

RESPONSE HISTORY ANALYSIS ($\ddot{u}_g(t)$ known)

Solve numerically $\ddot{q}_n + 2\xi \omega_n \dot{q}_n + \omega_n^2 q_n = -\Gamma_n \ddot{u}_g(t) \quad n = 1, 2, 3 \quad \rightarrow \quad q_n(t) \quad n = 1, 2, 3$

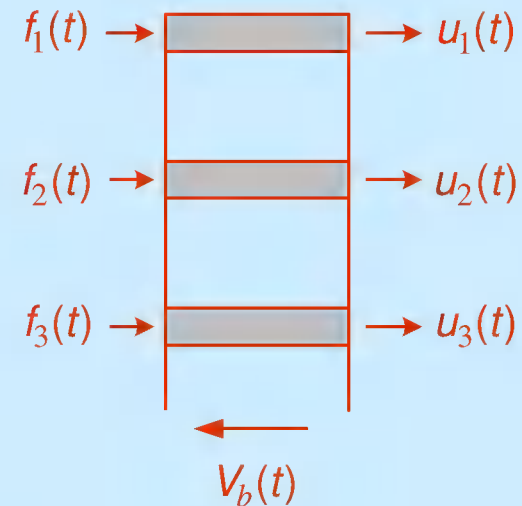
Summation of the modal contributions
$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = [\phi] \begin{Bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{Bmatrix} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} q_1(t) + \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{Bmatrix} q_2(t) + \begin{Bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{Bmatrix} q_3(t)$$

Equivalent static forces

$$\begin{Bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{Bmatrix} = [\mathbf{k}] \begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = [\mathbf{k}][\phi] \begin{Bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{Bmatrix}$$

$$\begin{Bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \end{Bmatrix} = [\mathbf{k}] \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} q_1(t) + [\mathbf{k}] \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{Bmatrix} q_2(t) + [\mathbf{k}] \begin{Bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{Bmatrix} q_3(t)$$

Observation : The contribution of the higher modes can often be neglected.



The internal forces and stresses can now be determined using statics.

Example : base shear force

$$\text{static} \quad V_b(t) = f_1(t) + f_2(t) + f_3(t) \quad V_{b\max} = \max |V_b(t)|$$

RESPONSE SPECTRUM ANALYSIS

($\ddot{u}_g(t)$ unknown)

The design response spectrum is used to estimate the behaviour of the shear building.

The modal transformation gave (page 7.7)

$$\ddot{q}_n + 2\xi_n \omega_n \dot{q}_n + \omega_n^2 q_n = -\Gamma_n \ddot{u}_g(t) \quad (1) \quad n = 1, 2, 3$$

which can be compared to the equation for a SDF (page 7.1)

$$\ddot{u} + 2\xi_n \omega_n \dot{u} + \omega_n^2 u = -\ddot{u}_g(t) \quad (2)$$

D_n , the maximum displacement for equation (2) is found on the design response spectrum.

The maximum displacement for equation (1) is

$$q_{n_{\max}} = \Gamma_n D_n$$

Summation of the modal contributions

$$\begin{Bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{Bmatrix} = [\phi] \begin{Bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{Bmatrix}$$

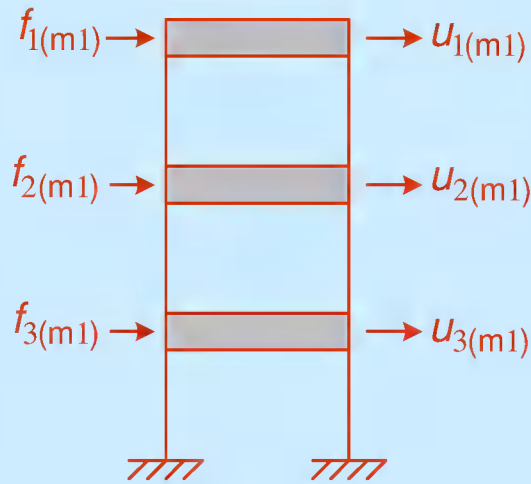
$$\text{but} \quad \begin{Bmatrix} u_{1\max} \\ u_{2\max} \\ u_{3\max} \end{Bmatrix} \neq [\phi] \begin{Bmatrix} \Gamma_1 D_1 \\ \Gamma_2 D_2 \\ \Gamma_3 D_3 \end{Bmatrix} \quad (3)$$

because D_1, D_2, D_3 do not occur at the same time.

(3) over estimates the maximal real displacements.

The summation of modal contributions is performed in a different way.

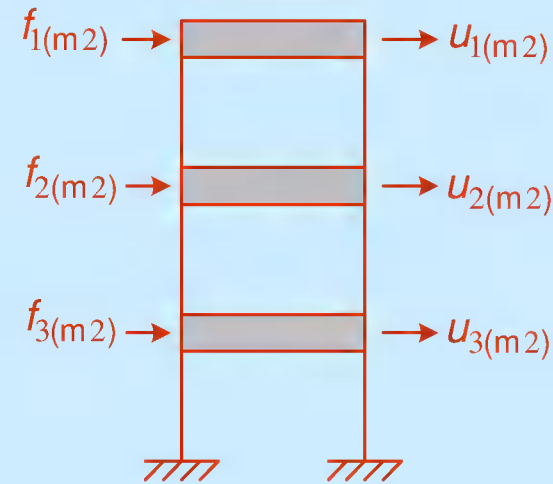
The idea is to calculate first maximum displacements and equivalent static forces for each mode.



MODE 1

$$\begin{Bmatrix} u_{1(m1)} \\ u_{2(m1)} \\ u_{3(m1)} \end{Bmatrix} = [\phi] \begin{Bmatrix} \Gamma_1 D_1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} \Gamma_1 D_1$$

$$\begin{Bmatrix} f_{1(m1)} \\ f_{2(m1)} \\ f_{3(m1)} \end{Bmatrix} = [\mathbf{k}][\phi] \begin{Bmatrix} \Gamma_1 D_1 \\ 0 \\ 0 \end{Bmatrix} = [\mathbf{k}] \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} \Gamma_1 D_1$$



MODE 2

$$\begin{Bmatrix} u_{1(m2)} \\ u_{2(m2)} \\ u_{3(m2)} \end{Bmatrix} = [\phi] \begin{Bmatrix} 0 \\ \Gamma_2 D_2 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{Bmatrix} \Gamma_2 D_2$$

$$\begin{Bmatrix} f_{1(m2)} \\ f_{2(m2)} \\ f_{3(m2)} \end{Bmatrix} = [\mathbf{k}][\phi] \begin{Bmatrix} 0 \\ \Gamma_2 D_2 \\ 0 \end{Bmatrix} = [\mathbf{k}] \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{Bmatrix} \Gamma_2 D_2$$

Same calculations for the third mode

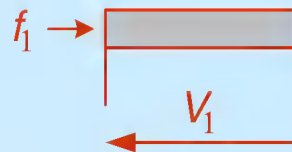
Once a value of the required response has been calculated for each mode separately, the superposition of the different modes is performed using an approximative rule. The most common one is the SRSS approximation.

Maximum displacements

$$\begin{Bmatrix} u_{1\max} \\ u_{2\max} \\ u_{3\max} \end{Bmatrix} = \begin{Bmatrix} \sqrt{u_{1(m1)}^2 + u_{1(m2)}^2 + u_{1(m3)}^2} \\ \sqrt{u_{2(m1)}^2 + u_{2(m2)}^2 + u_{2(m3)}^2} \\ \sqrt{u_{3(m1)}^2 + u_{3(m2)}^2 + u_{3(m3)}^2} \end{Bmatrix}$$

Maximum shear force in the highest floor

$$V_{1\max} = \sqrt{f_{1(m1)}^2 + f_{1(m2)}^2 + f_{1(m3)}^2}$$



Maximum base shear force

Maximum base shear force due to the mode 1

$$V_{b(m1)} = f_{1(m1)} + f_{2(m1)} + f_{3(m1)}$$

Maximum base shear force due to the mode 2

$$V_{b(m2)} = f_{1(m2)} + f_{2(m2)} + f_{3(m2)}$$

Maximum base shear force due to the mode 3

$$V_{b(m3)} = f_{1(m3)} + f_{2(m3)} + f_{3(m3)}$$

Total maximum base shear force

$$V_{b\max} = \sqrt{V_{b(m1)}^2 + V_{b(m2)}^2 + V_{b(m3)}^2}$$

Observation : The contribution of the higher modes can often be neglected.

FINITE ELEMENT METHOD

The equation system obtained by f.e.m. is

$$[\mathbf{m}]\{\ddot{\mathbf{u}}\} + [\mathbf{c}]\{\dot{\mathbf{u}}\} + [\mathbf{k}]\{\mathbf{u}\} = \{\mathbf{p}(t)\}$$

with

$[\mathbf{m}]$: mass matrix

$[\mathbf{c}]$: damping matrix

$[\mathbf{k}]$: stiffness matrix

$\{\mathbf{p}(t)\}$: generalised load vector

$\{\mathbf{u}(t)\}$: displacement vector

The generalised load vector and stiffness matrix have been studied in the f.e.m. course and are identical in static and dynamic analyses.

The damping matrix is not obtained by finite element discretisation. The different ways of introducing and dealing with damping have been studied in lesson 6.

The resolution of the equation system, by mode superposition or by direct time integration methods has been studied in lessons 4 and 6.

The only remaining work is to present the derivation of the element mass matrix. The assembly of the element mass matrices to the structure mass matrix is performed in the same way as for the stiffness matrix.

CONSISTENT ELEMENT MASS MATRIX

The equation of motion can be derived by equating the work done by the externally applied loads (external work) with the work absorbed by inertial and dissipative forces (internal work) for any virtual displacement (that is, for any imagined small motions that satisfies compatibility and essential boundary conditions).

$$W_i = \int_V \left(\{\delta \boldsymbol{\varepsilon}\}^T \{\boldsymbol{\sigma}\} + \{\delta \mathbf{u}\}^T \rho \{\ddot{\mathbf{u}}\} \right) dv$$

$$W_e = \int_V \{\delta \mathbf{u}\}^T \{\mathbf{F}\} dv + \int_S \{\delta \mathbf{u}\}^T \{\Phi\} ds + \sum_{i=1}^n \{\delta \mathbf{u}\}_i^T \{\mathbf{f}\}_i$$

where $\{\mathbf{F}\}$ and $\{\Phi\}$ are the prescribed body forces and surface tractions, $\{\mathbf{f}\}_i$ and $\{\delta \mathbf{u}\}_i$ represent prescribed concentrated loads and their corresponding virtual displacements at a total of n points, ρ is the mass density.

$\{\delta \mathbf{u}\}$ and $\{\delta \boldsymbol{\varepsilon}\}$ represent virtual displacements and their corresponding strains.

Finite element discretisation provides

$$\{\mathbf{u}\} = [\mathbf{N}]\{\mathbf{d}\} \quad \{\ddot{\mathbf{u}}\} = [\mathbf{N}]\{\ddot{\mathbf{d}}\} \quad \{\boldsymbol{\varepsilon}\} = [\mathbf{B}]\{\mathbf{d}\}$$

Shape functions $[\mathbf{N}]$ are functions of space while nodal d.o.f. $\{\mathbf{d}\}$ are function of time.

A linear elastic material is assumed

$$\{\boldsymbol{\sigma}\} = [\mathbf{E}]\{\boldsymbol{\varepsilon}\}$$

The internal and external virtual works can then be rewritten as

$$W_i = \{\delta \mathbf{d}\}^T \left[\int_V \left([\mathbf{B}]^T [\mathbf{E}] [\mathbf{B}] \{\mathbf{d}\} + [\mathbf{N}]^T \rho [\mathbf{N}] \{\ddot{\mathbf{d}}\} \right) dv \right]$$

$$W_e = \{\delta \mathbf{d}\}^T \left[\int_V [\mathbf{N}]^T \{\mathbf{F}\} dv + \int_S [\mathbf{N}]^T \{\Phi\} ds + \sum_{i=1}^n [\mathbf{N}]^T \{\mathbf{f}\}_i \right]$$

The last equations in the preceding page are

$$W_i = \{\delta \mathbf{d}\}^T \left[\int_V \left([\mathbf{B}]^T [\mathbf{E}] [\mathbf{B}] \{\mathbf{d}\} + [\mathbf{N}]^T \rho [\mathbf{N}] \{\ddot{\mathbf{d}}\} \right) dV \right]$$

$$W_e = \{\delta \mathbf{d}\}^T \left[\int_V [\mathbf{N}]^T \{\mathbf{F}\} dV + \int_s [\mathbf{N}]^T \{\Phi\} ds + \sum_{i=1}^n [\mathbf{N}]^T \{\mathbf{f}\}_i \right]$$

$W_i = W_e$ is true for arbitrary $\{\delta \mathbf{d}\}$, which gives

$$\int_V \rho [\mathbf{N}]^T [\mathbf{N}] dV \{\ddot{\mathbf{d}}\} + \int_V [\mathbf{B}]^T [\mathbf{E}] [\mathbf{B}] dV \{\mathbf{d}\} = \{\mathbf{p}\}$$

$$\{\mathbf{p}\} = \int_V [\mathbf{N}]^T \{\mathbf{F}\} dV + \int_s [\mathbf{N}]^T \{\Phi\} ds + \sum_{i=1}^n [\mathbf{N}]^T \{\mathbf{f}\}_i$$

This can be rewritten as

$$[\mathbf{m}] \{\ddot{\mathbf{u}}\} + [\mathbf{k}] \{\mathbf{u}\} = \{\mathbf{p}(t)\}$$

with

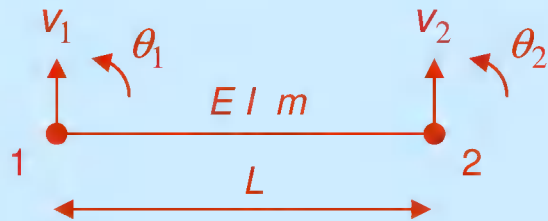
$$[\mathbf{k}] = \int_V [\mathbf{B}]^T [\mathbf{E}] [\mathbf{B}] dV$$

$$[\mathbf{m}] = \int_V \rho [\mathbf{N}]^T [\mathbf{N}] dV$$

The mass matrix defined in the last equation is called **consistent element mass matrix**.

The word consistent emphasizes that this form has been derived using the same shape functions as the element stiffness matrix.

BERNOULLI 2D BEAM ELEMENT



$$m = \rho A$$

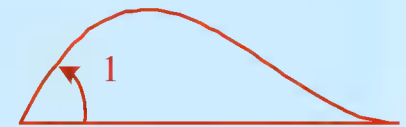
$$v(x,t) = [\mathbf{N}] \{\mathbf{d}\} \quad \{\mathbf{d}\} = \begin{Bmatrix} v_1(t) \\ \theta_1(t) \\ v_2(t) \\ \theta_2(t) \end{Bmatrix}$$

$$[\mathbf{N}] = [\varphi_1(x) \ \varphi_2(x) \ \varphi_3(x) \ \varphi_4(x)]$$

shape functions



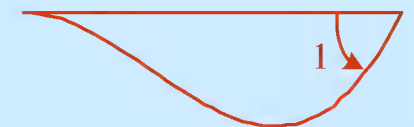
$$\varphi_1(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3$$



$$\varphi_2(x) = x\left(1 - \frac{x}{L}\right)^2$$



$$\varphi_3(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3$$



$$\varphi_4(x) = \frac{x^2}{L}\left(\frac{x}{L} - 1\right)$$

$$m_{ij} = \int_0^L m \varphi_i(x) \varphi_j(x) dx$$

$$[\mathbf{m}] = \frac{mL}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

$$[\mathbf{k}] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

BERNOULLI 2D BEAM ELEMENT WITH AXIAL DEFORMATIONS



A bar element is superposed to the previous beam element.

$$u(x,t) = \varphi_1(x) u_1 + \varphi_2(x) u_2$$

The shape functions for the bar element are

$$\varphi_1(x) = 1 - \frac{x}{L} \quad \varphi_2(x) = \frac{x}{L}$$

$$[\mathbf{m}] = \frac{mL}{420} \begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22L & 0 & 54 & -13L \\ 0 & 22L & 4L^2 & 0 & 13L & -3L^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & -22L \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix}$$

$$[\mathbf{k}] = \begin{bmatrix} EA/L & 0 & 0 & -EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\ 0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\ -EA/L & 0 & 0 & EA/L & 0 & 0 \\ 0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\ 0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L \end{bmatrix}$$

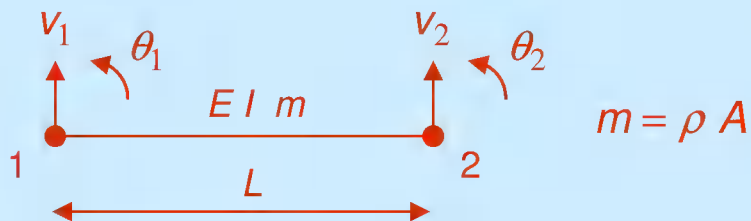
LUMPED MASS MATRIX

For certain applications it is better to use a diagonal (or lumped) mass matrix.

Two obvious advantages are less storage space and less processing time, especially in case of an explicit time integration scheme.

PARTICLE MASS LUMPING

Example : Bernoulli 2D beam element

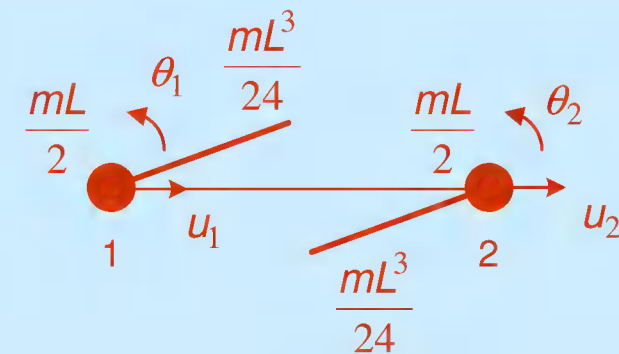


An easy way of obtaining a lumped mass matrix is to replace the distributed mass by two particles of mass $mL/2$ at each node.

The rotational inertia is defined by considering that a uniform slender bar of length $L/2$ and mass $mL/2$ is attached at each node.

The associate inertia moment is

$$J = (mL/2)(L/2)^2/3$$



$$[\mathbf{m}] = \frac{mL}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & L^2/12 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & L^2/12 \end{bmatrix}$$

HRZ LUMPING

Different methods can be used to transform the consistent element mass matrix and obtain a diagonal matrix. One of them is the HRZ lumping.

The idea is to compute only diagonal terms of the consistent element mass matrix, then scale them so as to preserve the total element mass. Since there may be both translational and rotational d.o.f., the method for an element of total mass M is

1. Compute diagonal coefficients m_{ij} of the consistent element mass matrix.
2. For each coordinate direction in which motion is described by the element d.o.f.
 - a. determine a number S by adding the m_{ij} associated with translational d.o.f. only.
 - b. multiply all coefficients m_{ij} associated with this direction by the ratio M/S .

Example : Bernoulli 2D beam element

1.

$$[\mathbf{m}] = \frac{mL}{420} \begin{bmatrix} 156 & 0 & 0 & 0 \\ 0 & 4L^2 & 0 & 0 \\ 0 & 0 & 156 & 0 \\ 0 & 0 & 0 & 4L^2 \end{bmatrix}$$

2.a.

$$S = \frac{312}{420} mL$$

$$M = mL$$

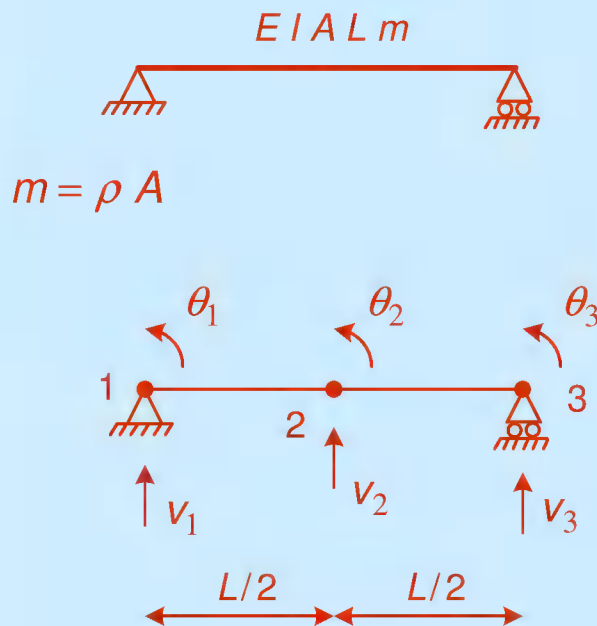
2.b.

$$[\mathbf{m}] = \frac{mL}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & L^2/39 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & L^2/39 \end{bmatrix}$$

EXAMPLE

The natural frequencies and eigenmodes of a simply supported beam are studied.

The beam is discretised in 2 elements.



Mass and stiffness matrices for the structure

$$L_e = L/2$$

$$[k] = \frac{EI}{L_e^3} \begin{bmatrix} 12 & 6L_e & -12 & 6L_e & 0 & 0 \\ 6L_e & 4L_e^2 & -6L_e & 2L_e^2 & 0 & 0 \\ -12 & -6L_e & 12+12 & -6L_e+6L_e & -12 & 6L_e \\ 6L_e & 2L_e^2 & -6L_e+6L_e & 4L_e^2+4L_e^2 & -6L_e & 2L_e^2 \\ 0 & 0 & -12 & -6L_e & 12 & -6L_e \\ 0 & 0 & 6L_e & 2L_e^2 & -6L_e & 4L_e^2 \end{bmatrix}$$

$$[m] = \frac{mL_e}{420} \begin{bmatrix} 156 & 22L_e & 54 & -13L_e & 0 & 0 \\ 22L_e & 4L_e^2 & 13L_e & -3L_e^2 & 0 & 0 \\ 54 & 13L_e & 156+156 & -22L_e+22L_e & 54 & -13L_e \\ -13L_e & -3L_e^2 & -22L_e+22L_e & 4L_e^2+4L_e^2 & 13L_e & -3L_e^2 \\ 0 & 0 & 54 & 13L_e & 156 & -22L_e \\ 0 & 0 & -13L_e & -3L_e^2 & -22L_e & 4L_e^2 \end{bmatrix}$$

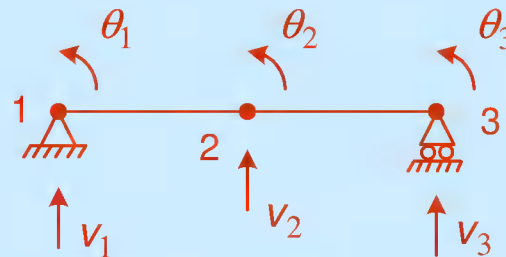
Boundary conditions

$$v_1 = v_3 = 0$$

$$\ddot{v}_1 = \ddot{v}_3 = 0$$

$p_1(t), p_3(t)$ unknowns

$$\begin{bmatrix} \ddot{v}_1 \\ \ddot{\theta}_1 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \\ \ddot{v}_3 \\ \ddot{\theta}_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} p_1(t) \\ 0 \\ 0 \\ 0 \\ p_3(t) \\ 0 \end{bmatrix}$$



Equation system

$$\frac{mL}{420} \begin{bmatrix} 4L_e^2 & 13L_e & -3L_e^2 & 0 \\ 13L_e & 312 & 0 & -13L_e \\ -3L_e^2 & 0 & 8L_e^2 & -3L_e^2 \\ 0 & -13L_e & -3L_e^2 & 4L_e^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{v}_2 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \frac{EI}{L^3} \begin{bmatrix} 4L_e^2 & -6L_e & 2L_e^2 & 0 \\ -6L_e & 24 & 0 & 6L_e \\ 2L_e^2 & 0 & 8L_e^2 & 2L_e^2 \\ 0 & 6L_e & 2L_e^2 & 4L_e^2 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Solution of the eigenvalue problem (exact values in parenthesis)

$$\omega_1 = 9.91 \sqrt{\frac{EI}{mL^4}} \quad (9.87)$$

$$\omega_2 = 43.8 \sqrt{\frac{EI}{mL^4}} \quad (39.5)$$

$$\omega_3 = 110 \sqrt{\frac{EI}{mL^4}} \quad (88.9)$$

$$\omega_4 = 201 \sqrt{\frac{EI}{mL^4}} \quad (158)$$

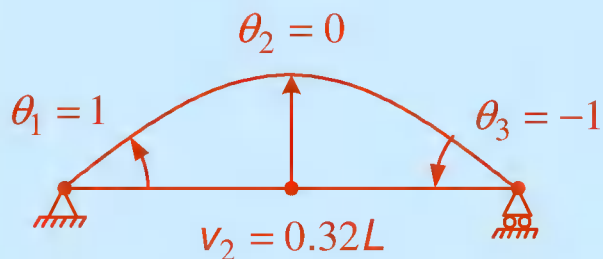
$$\phi_1 = \begin{Bmatrix} 1 \\ 0.32 L \\ 0 \\ -1 \end{Bmatrix}$$

$$\phi_2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{Bmatrix}$$

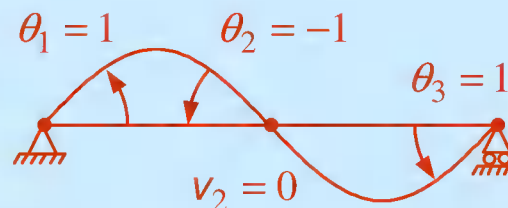
$$\phi_3 = \begin{Bmatrix} 1 \\ -0.05 L \\ 0 \\ -1 \end{Bmatrix}$$

$$\phi_4 = \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{Bmatrix}$$

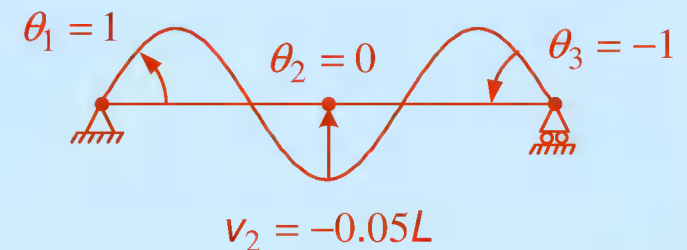
eigenmode 1



eigenmode 2

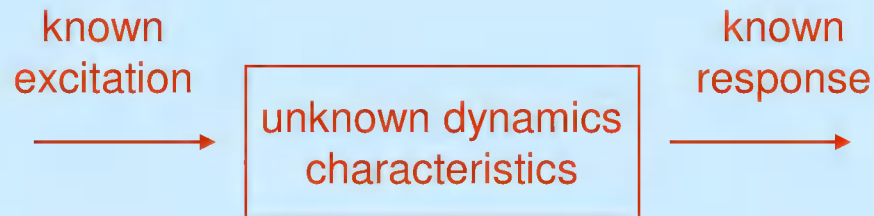


eigenmode 3



EXPERIMENTAL DYNAMICS

The purpose of experimental dynamics is to determine by measurements the dynamics characteristics of a structure. This is performed by measuring the response of the structure to a known excitation.



Unknown dynamics characteristics

SDF : natural frequency f_n
 damping coefficient ξ

MDF : natural frequencies $f_1 \ f_2 \ f_3 \ \dots$
 eigenmodes $\phi_1 \ \phi_2 \ \phi_3 \ \dots$
 modes damping $\xi_1 \ \xi_2 \ \xi_3 \ \dots$

Different types of excitations can be used

Free vibration : the structure is disturbed from equilibrium and then vibrates without any applied forces.

Harmonic force : a sinusoidal force is applied through a special device.

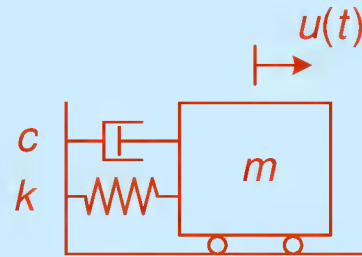
Pulse force : a hammer is used to apply a short pulse force.

Random force : random noise as e.g. traffic or wind loads are applied.

For practical reasons, accelerations are easier to measure than displacements

SDF SYSTEMS

Objective : determine experimentally f_n and ξ

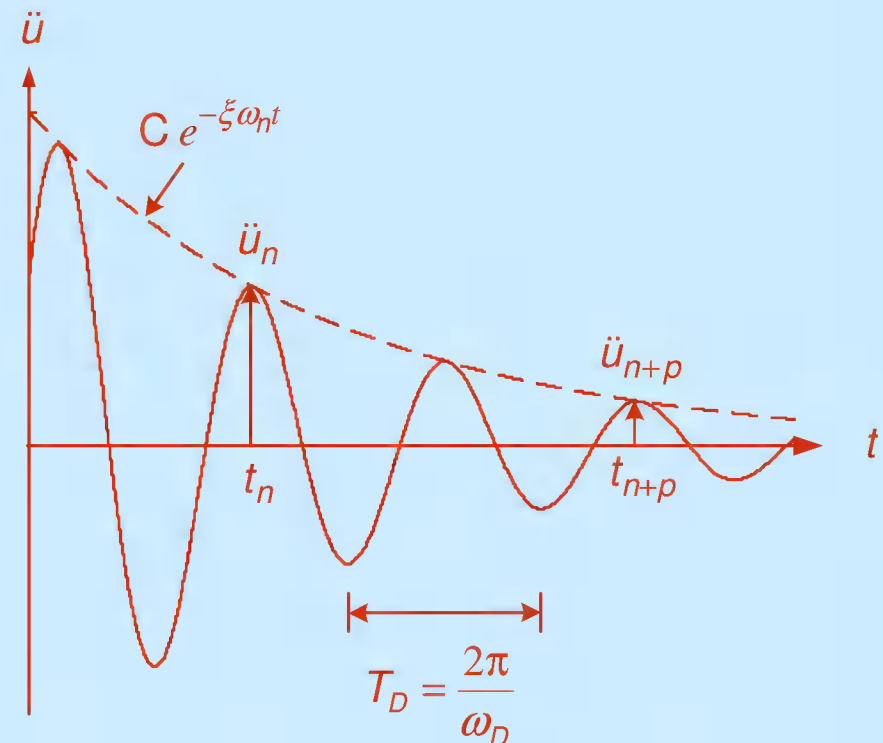


Free vibration test

$$u(t) = C e^{-\xi \omega_n t} \sin(\omega_D t + \theta)$$

$$\rightarrow \ddot{u}(t) = D e^{-\xi \omega_n t} \sin(\omega_D t + \alpha)$$

After derivation a similar expression for the acceleration is obtained and consequently the results derived in Lesson 1 for the displacements can be used.



$$\begin{aligned} \xi < 0.1 \quad \rightarrow \quad \xi &= \frac{1}{2\pi p} \ln \frac{\ddot{u}_n}{\ddot{u}_{n+p}} \\ \rightarrow \quad f_n &= \frac{1}{T_n} \approx \frac{1}{T_D} \end{aligned}$$

Harmonic force test

The structure is excited by a harmonic force whose frequency is slowly increased step by step in order to reach the steady state response at each increment. The amplitudes of accelerations are measured at each step.

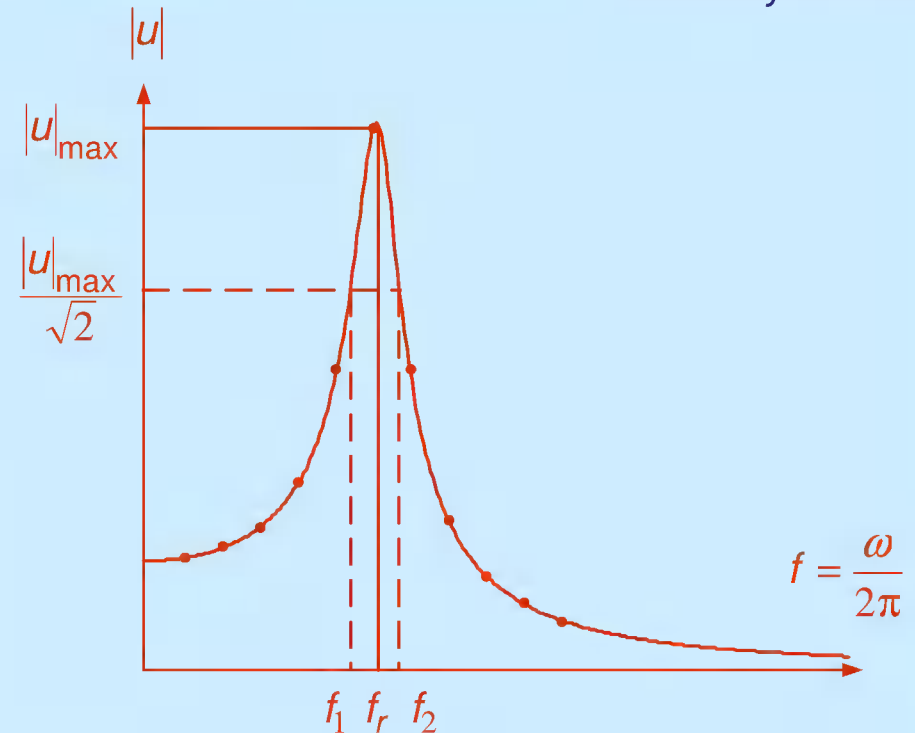
Results from Lesson 2 :

$$p(t) = p_o \sin(\omega t) \quad u = |u| \sin(\omega t + \theta)$$

$$|u| = \frac{p_o/k}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\xi(\omega/\omega_n)]^2}}$$

By derivation, it is obtained :

$$\ddot{u} = -|\ddot{u}| \sin(\omega t + \theta) \quad |\ddot{u}| = \omega^2 |u|$$



$$\xi < 0.1 \rightarrow f_n = f_r \quad \xi = \frac{f_2 - f_1}{f_2 + f_1}$$

In reality, the maximal value is not known and a curve fitting is done.

If the damping is low, the same method can be used with $|\ddot{u}|$ instead of $|u|$.

Ambient vibration test

The structure is excited by a small random force, which means that the force is not known.

This method is often used for bridges where wind and traffic loads are used as random forces.

$$\ddot{u}(t) = C e^{-\xi \omega_n t} \sin(\omega_d t + \alpha) + \text{Particular solution}$$

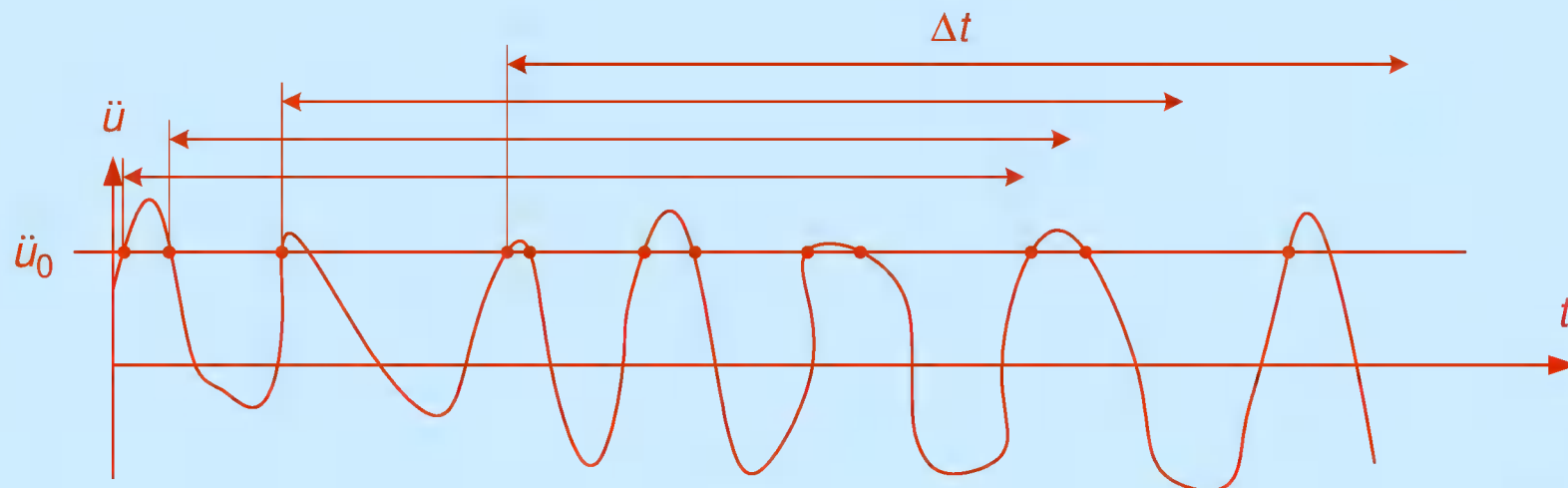
initial conditions $\ddot{u}_0$ $\dot{u}_0$

random load

The response (acceleration) is the sum of the free vibration solution and the particular solution which depends on the random load.

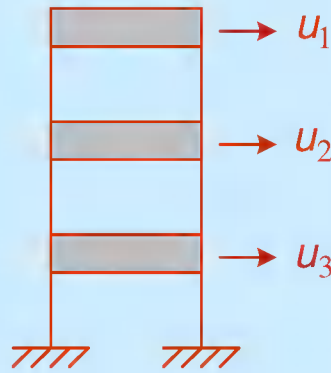
The idea of this approach is to remove the particular solution and the initial condition $\dot{u}_0$ by taking the average of many subrecords of same length Δt and same $\ddot{u}_0$. The result is a free vibration solution which can be then used.

The trigger value $\ddot{u}_0$ and the length Δt must be carefully chosen.



MDF SYSTEMS

Objective : determine
experimentally $f_1 f_2 f_3$
 $\phi_1 \phi_2 \phi_3$ and $\xi_1 \xi_2 \xi_3$



Harmonic force test

The structure is excited by a harmonic force whose frequency is slowly increased step by step in order to reach the steady state response at each increment. The amplitudes of accelerations are measured at each step.

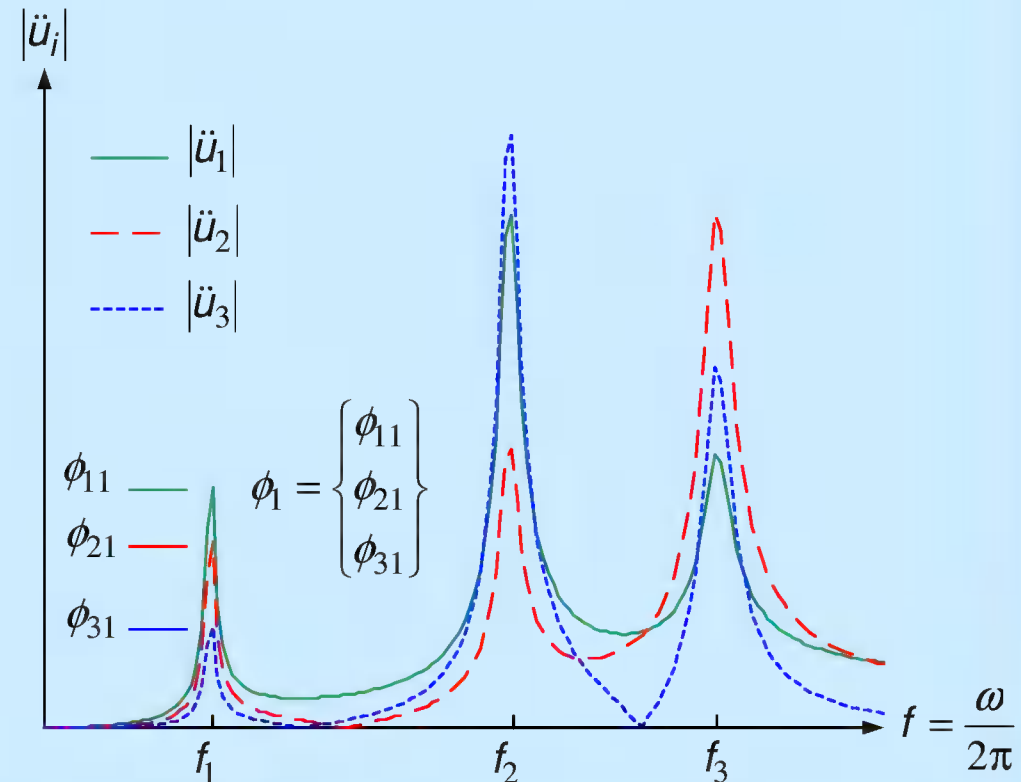
Results from Lesson 6 :

$$\begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \end{Bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} \sin(\omega t + \theta)$$

$$\omega = \omega_1 \quad (f = f_1)$$

$$\begin{matrix} A_1 \text{ large} \\ A_2, A_3 \text{ negligible} \end{matrix} \rightarrow \begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \end{Bmatrix} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} A_1 \sin(\omega t + \theta)$$

The structure vibrates with a deflected shape corresponding to eigenmode 1 and the amplitudes of vibrations are large : resonance.



The sign of ϕ_{ij} is determined by the phase which is also measured.

Explication : the solution at the first resonance is e.g.
$$\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} = C \begin{Bmatrix} 1.00 \\ -0.55 \\ -1.25 \end{Bmatrix} \sin(\omega t + 30^\circ)$$

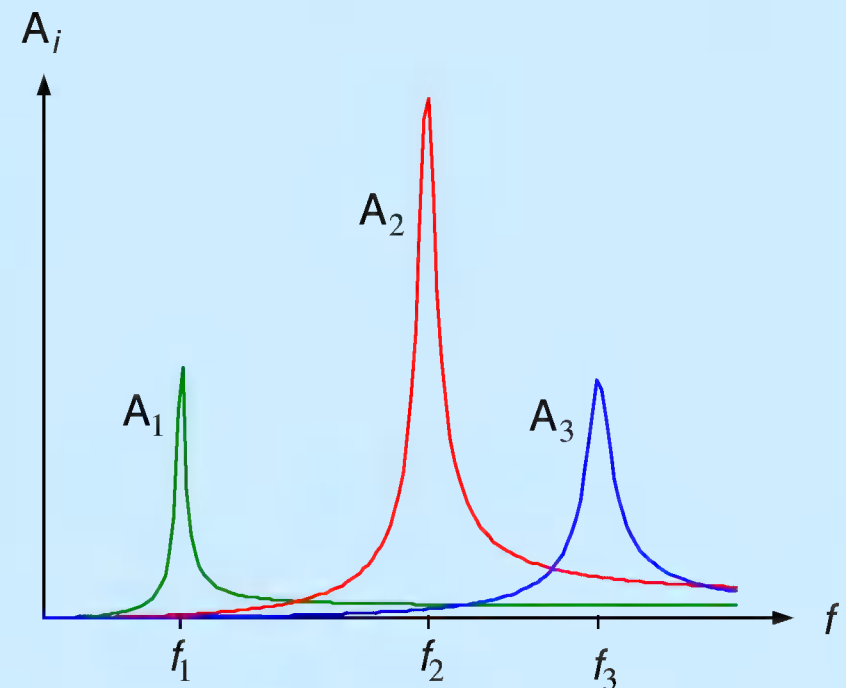
which can be rewritten as
$$\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} = C \begin{Bmatrix} 1.00 \sin(\omega t + 30^\circ) \\ 0.55 \sin(\omega t + 30^\circ + 180^\circ) \\ 1.25 \sin(\omega t + 30^\circ + 180^\circ) \end{Bmatrix}$$

The measured phase is either 30° or 210° , which gives the sign of ϕ_{ij} .

ξ_1, ξ_2, ξ_3 are determined by band width method for each resonance.

Remark : this method assumes that only one mode is acting at each resonance. For that, the natural frequencies must be well separated:

$\omega = \omega_i \rightarrow A_i$ large and other A_j negligible



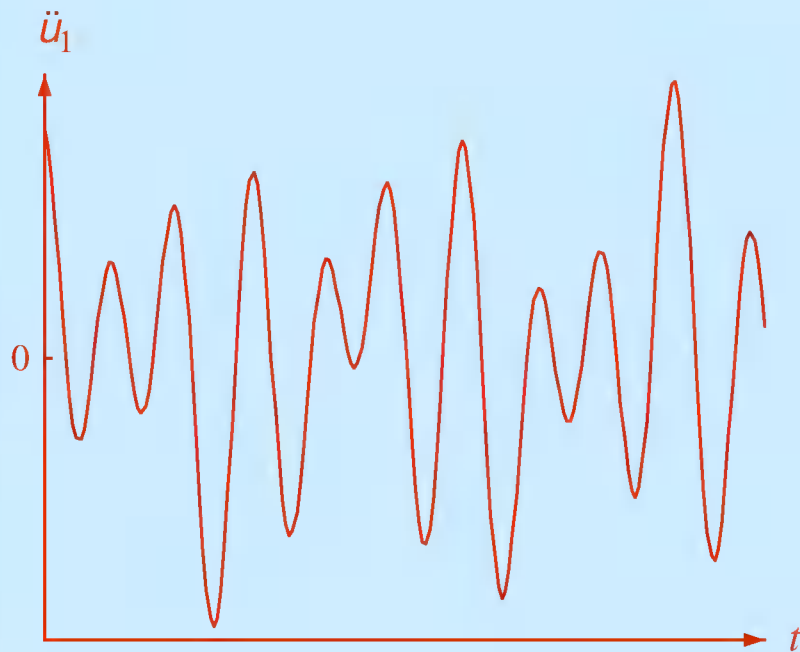
In that case, the influence of the 2nd eigenmode can not be neglected at the 3rd resonance.

Free vibration test (damping neglected)

Results from Lesson 5 :

$$\begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \end{Bmatrix} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} \sin(\omega_1 t + \theta_1) + \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{Bmatrix} \sin(\omega_2 t + \theta_2) + \begin{Bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{Bmatrix} \sin(\omega_3 t + \theta_3)$$

$$\ddot{u}_1(t) = \phi_{11} \sin(\omega_1 t + \theta_1) + \phi_{12} \sin(\omega_2 t + \theta_2) + \phi_{13} \sin(\omega_3 t + \theta_3)$$

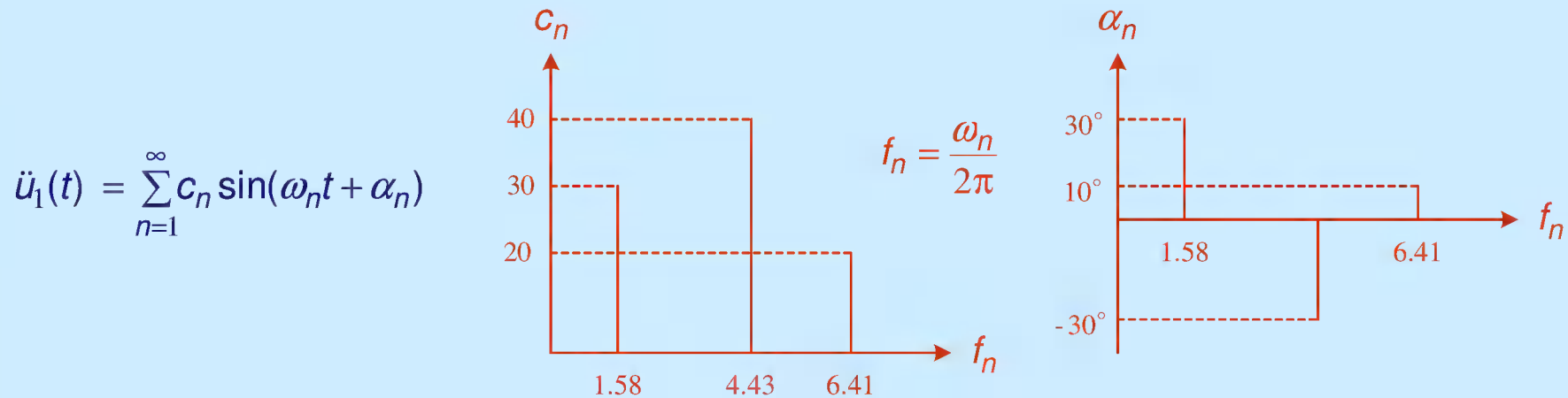


It is impossible to identify the dynamics parameters ϕ_{11} ϕ_{12} ϕ_{13} ω_1 ω_2 ω_3 θ_1 θ_2 θ_3 directly from the record of the acceleration of the first story.

The solution is to transform the record from the **time domain** to the **frequency domain** with a Fourier transformation:

$$\ddot{u}_1(t) = \sum_{n=1}^{\infty} [a_n \cos(\omega_n t) + b_n \sin(\omega_n t)] = \sum_{n=1}^{\infty} c_n \sin(\omega_n t + \alpha_n)$$

A numerical Fast Fourier transformation (FFT) gives the following two graphs, which are the representation of the signal in the **frequency domain**.



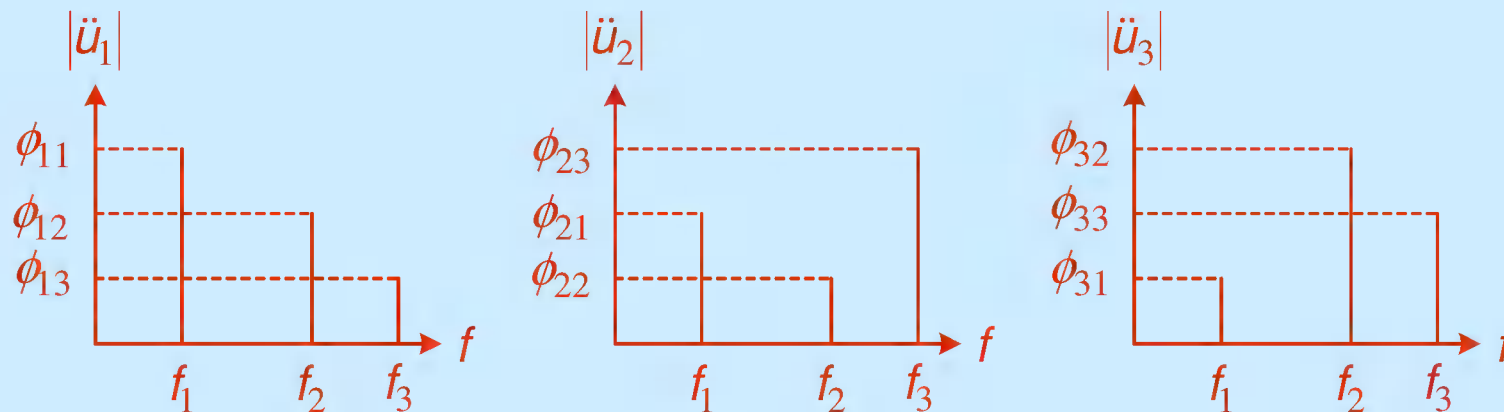
The required dynamics parameters can be easily determined from these two graphs:

$$\ddot{u}_1(t) = 30 \sin(2\pi \cdot 1.58t + 30^\circ) + 40 \sin(2\pi \cdot 4.43t - 30^\circ) + 20 \sin(2\pi \cdot 6.41t + 10^\circ)$$

$f_1 = 1.58 \text{ Hz}$	$\phi_{11} = 30$	$\theta_1 = 30^\circ$
$f_2 = 4.43 \text{ Hz}$	$\phi_{12} = 40$	$\theta_2 = -30^\circ$
$f_3 = 6.41 \text{ Hz}$	$\phi_{13} = 20$	$\theta_3 = 10^\circ$

The eigenmodes can be entirely determined by doing the same operation for the 2nd and 3rd story.

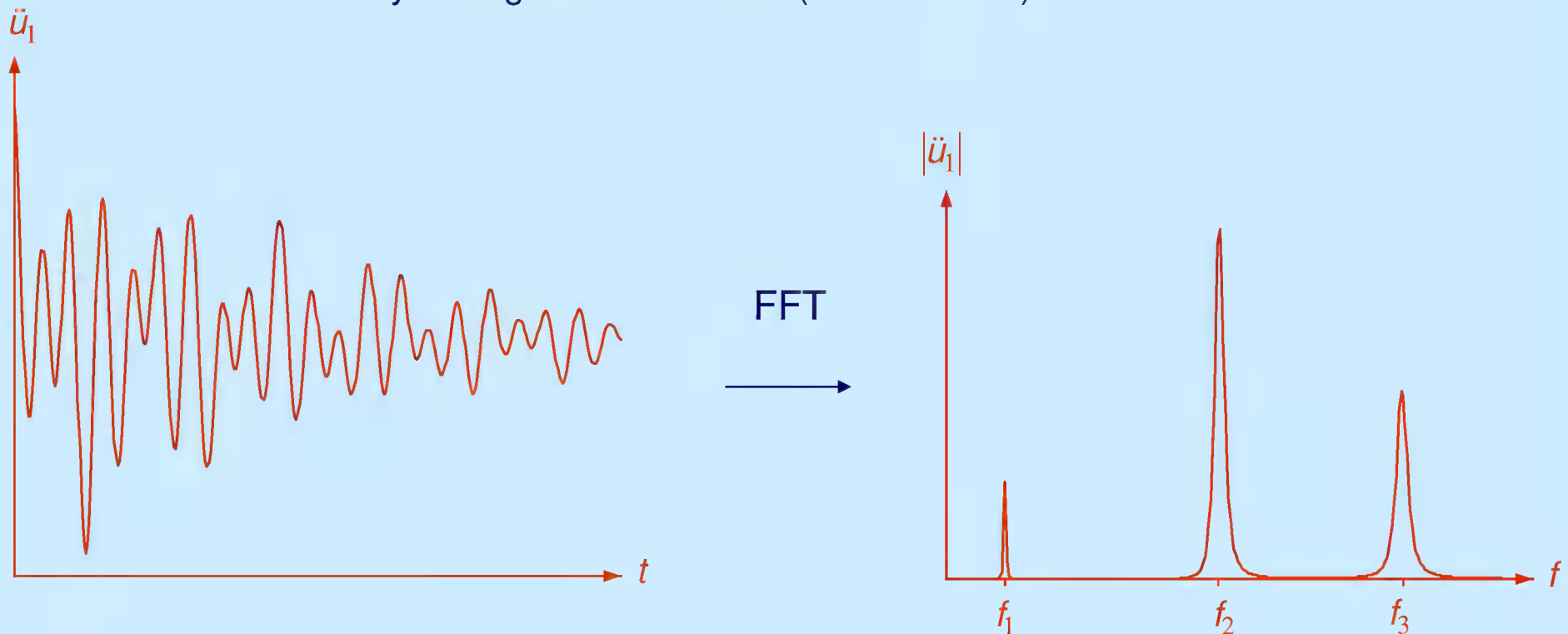
$$\begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \end{Bmatrix} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} \sin(\omega_1 t + \theta_1) + \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{Bmatrix} \sin(\omega_2 t + \theta_2) + \begin{Bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{Bmatrix} \sin(\omega_3 t + \theta_3)$$



The sign of each ϕ_{ij} is determined by the phase which is also given by a FFT.

Free vibration test (with damping)

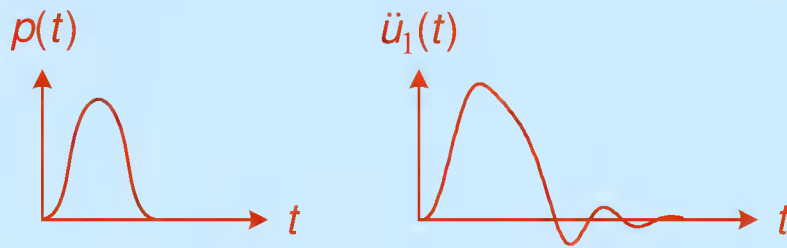
The same method is applied. The signal becomes periodic by artificially adding the same model (see Lesson 3)



ξ_1 , ξ_2 , ξ_3 are determined by band width method or by cutting the signal in the frequency domain and performing an inverse FFT to obtain free vibration of a SDF system corresponding to the eigenmode.

Pulse force excitation

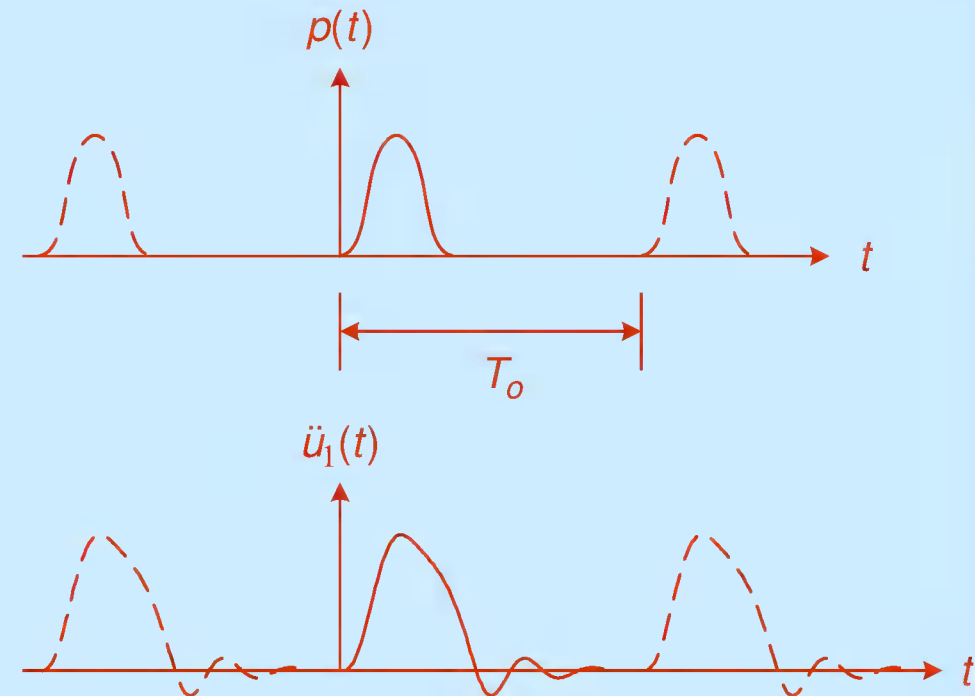
A hammer is used to apply a short pulse force.



$p(t)$ and the response are non periodic. They become periodic by artificially adding the same model. The artificial response is then the steady state response of the structure loaded by the artificial periodic load. The artificial load can be considered as the sum of harmonic terms. Each of these harmonic terms gives an harmonic response.

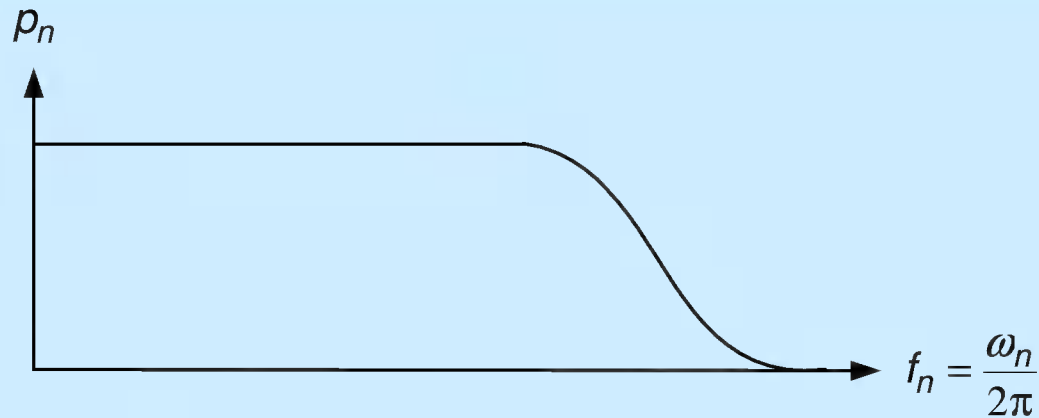
$$p_{\text{art}}(t) = \sum_{n=1}^{\infty} p_n \sin(\omega_n t + \alpha_n)$$

$$\begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \end{Bmatrix}_{\text{art}} = \sum_{n=1}^{\infty} \begin{Bmatrix} D_{1n} \\ D_{2n} \\ D_{3n} \end{Bmatrix} \sin(\omega_n t + \theta_1)$$



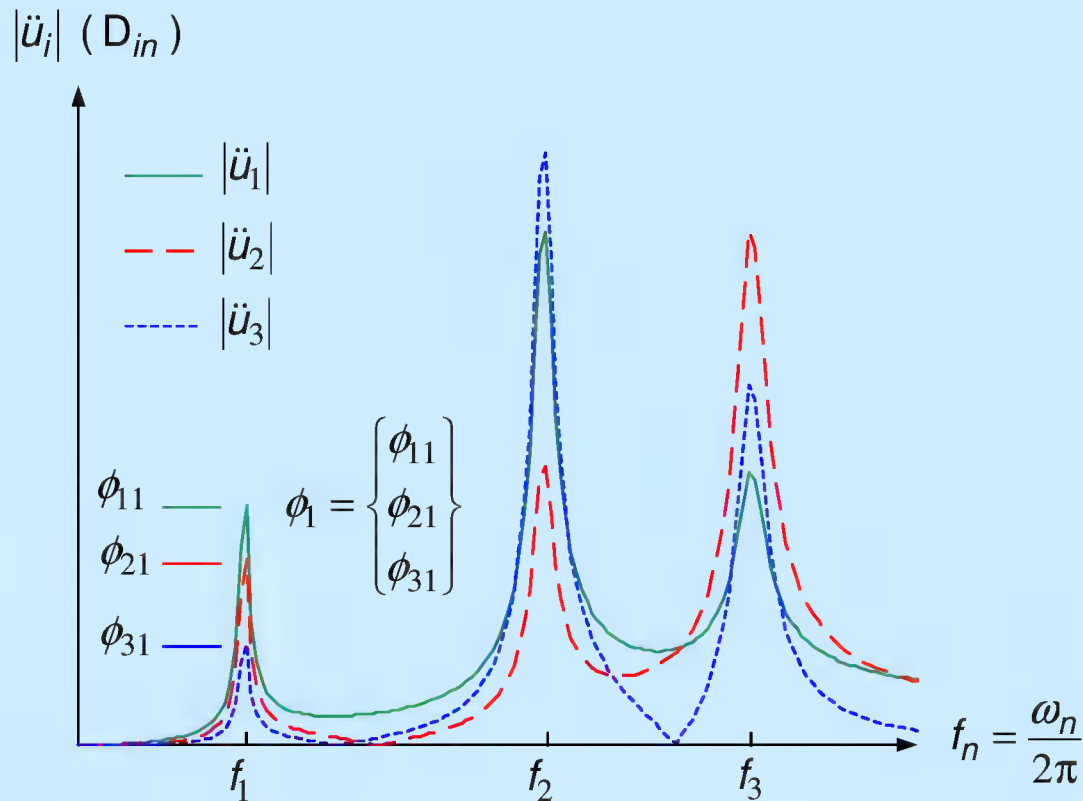
$$\omega_n = \omega_1 \rightarrow \text{resonance} \quad \begin{Bmatrix} D_{1n} \\ D_{2n} \\ D_{3n} \end{Bmatrix} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix}$$

The dynamic parameters can then be obtained by performing a FFT.



$$p_{\text{art}}(t) = \sum_{n=1}^{\infty} p_n \sin(\omega_n t + \alpha_n)$$

The load obtained by the hammer must contain all the frequencies that have to be studied.



$$\begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \\ \ddot{u}_3(t) \end{Bmatrix}_{\text{art}} = \sum_{n=1}^{\infty} \begin{Bmatrix} D_{1n} \\ D_{2n} \\ D_{3n} \end{Bmatrix} \sin(\omega_n t + \theta_1)$$

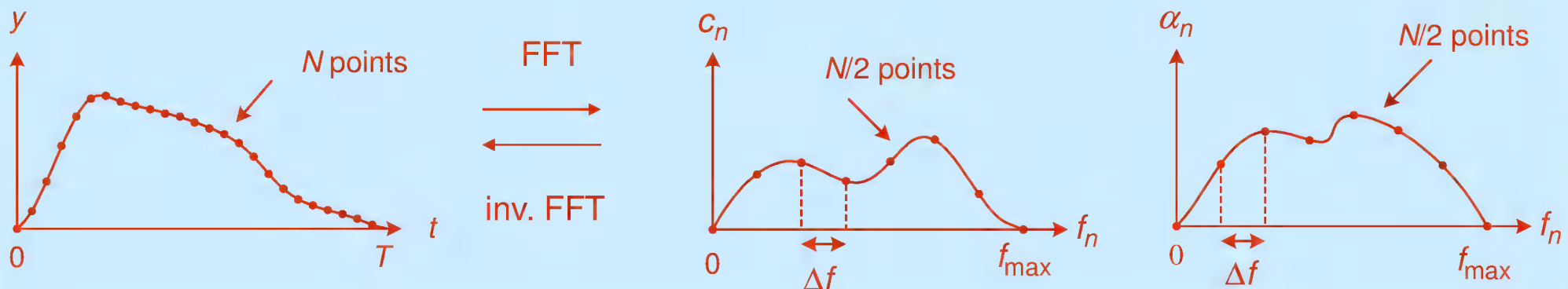
The FFT of the response is similar to the graph obtained with an harmonic force test (see page 9.5)

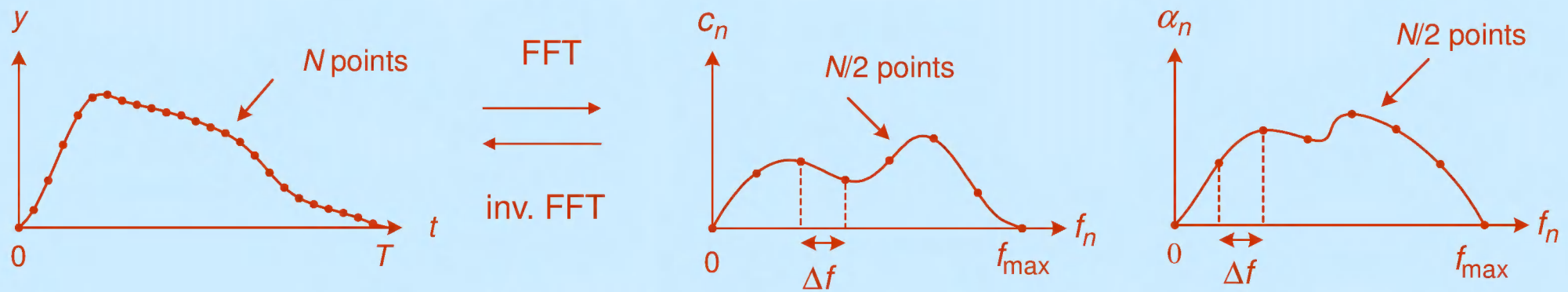
SIGNAL ANALYSIS - FFT

A signal can be represented in two different ways: in the **time domain** or in the **frequency domain**. Each domain has its own interest. The transformation between the two domains is performed numerically by a FFT or an inverse FFT.

$$\begin{array}{ccc}
 & \text{FFT} & \\
 y(t) & \xrightarrow{\quad} & Y(f) \begin{cases} a_n(f_n) \\ b_n(f_n) \end{cases} \text{ or } \begin{cases} c_n(f_n) \\ \alpha_n(f_n) \end{cases} \\
 & \xleftarrow{\text{inv. FFT}} & \\
 \end{array}
 \quad
 \begin{aligned}
 y(t) &= \sum_{n=1}^{\infty} [a_n \cos(2\pi f_n t) + b_n \sin(2\pi f_n t)] \\
 &= \sum_{n=1}^{\infty} c_n \sin(2\pi f_n t + \alpha_n)
 \end{aligned}$$

The signal in the frequency domain is represented by two graphs ($a_n b_n$) or ($c_n \alpha_n$).





$y(t)$ is defined numerically by N points during a time T .

The FFT requires $N = 2^m$ $m = 1, 2, 3, \dots$

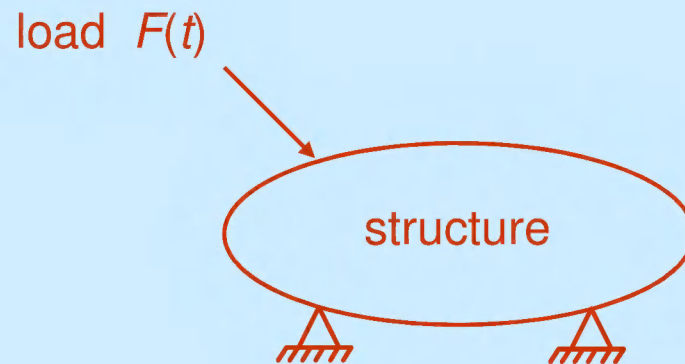
$$rate = \frac{N}{T}$$

c_n and α_n are defined by $N/2$ points.

$$\Delta_f = \frac{1}{T} \quad f_{\max} = \Delta_f \frac{N}{2} = \frac{rate}{2}$$

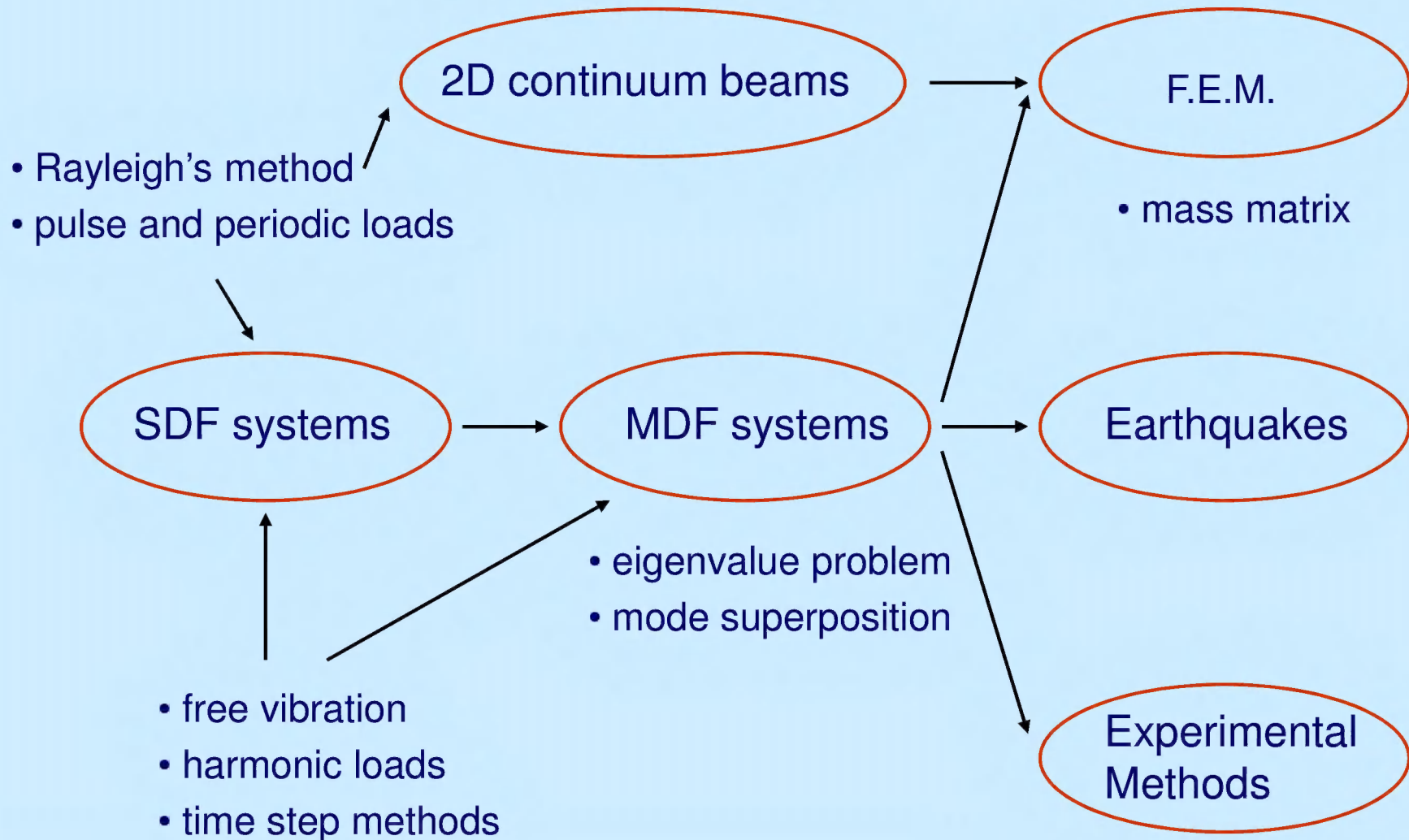
In practice, $(\Delta_f, f_{\max})$ are chosen and $(T, rate)$ are then determined such that $N = 2^m$.

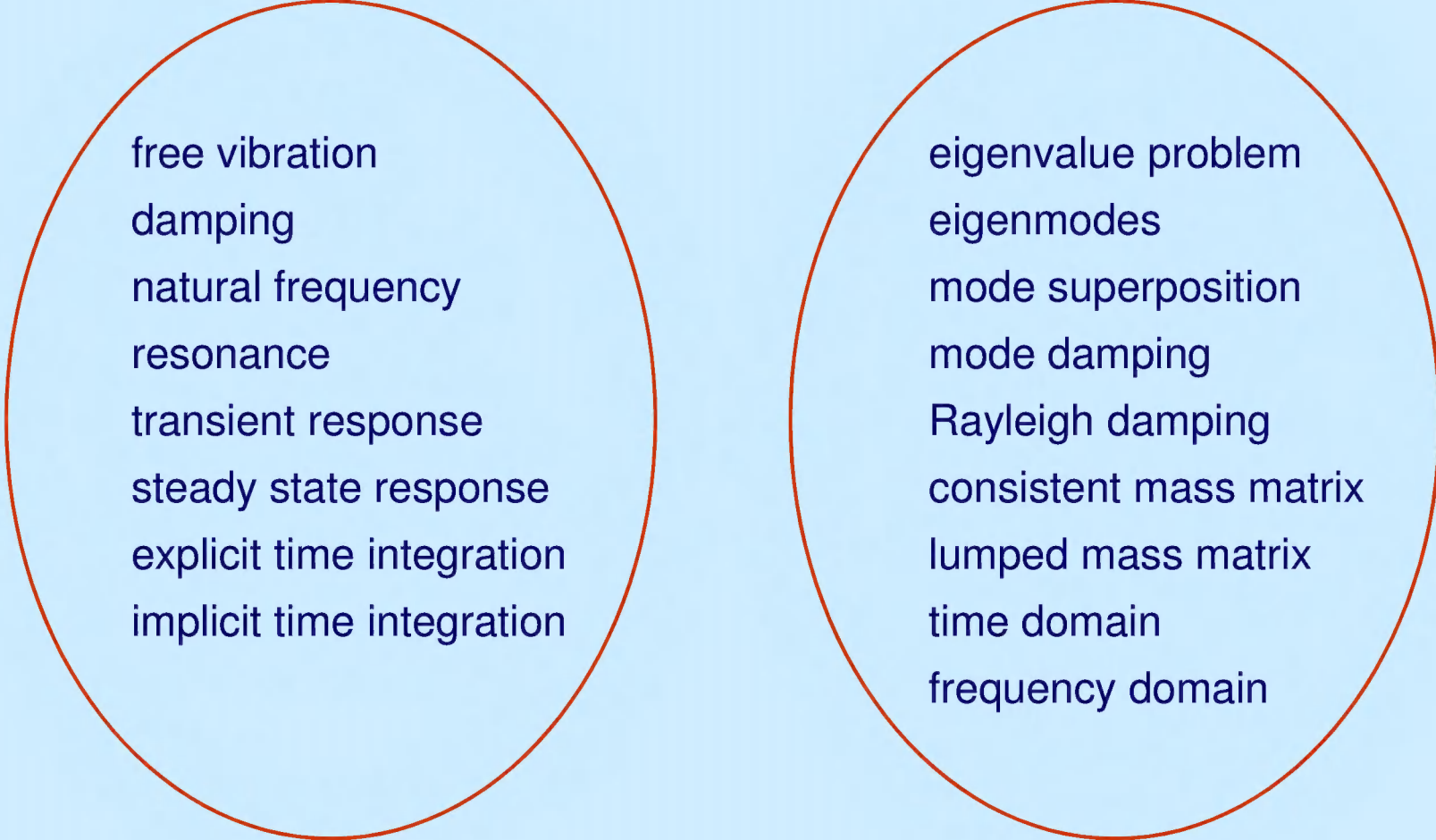
STRUCTURAL DYNAMICS – 5C1840



- displacements ?
- strains ?
- stresses ?

restriction : linear structures $m\ddot{u} + c\dot{u} + ku = F(t)$





free vibration
damping
natural frequency
resonance
transient response
steady state response
explicit time integration
implicit time integration

eigenvalue problem
eigenmodes
mode superposition
mode damping
Rayleigh damping
consistent mass matrix
lumped mass matrix
time domain
frequency domain